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Soliton spectrum of integrable models with local symmetries

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Abstract: The soliton spectrum (massive and massless) of a family of integrable models with local U(1) and U(1) $\otimes$ U(1) symmetries is studied. These models represent relevant integrable deformations of SL(2, $\mathbb{R}$) $\otimes$ U(1)$^{n-1}$-WZW and SL(2, $\mathbb{R}$) $\otimes$ SL(2, $\mathbb{R}$) $\otimes$ U(1)$^{n-2}$-WZW models. Their massless solitons appear as specific topological solutions of the U(1)- (or U(1) $\otimes$ U(1)-) CFTs. The nonconformal analog of the GKO-coset formula is derived and used in the construction of the composite massive solitons of the ungauged integrable models.

Keywords: Solitons Monopoles and Instantons, Conformal and W Symmetry, Integrable Hierarchies, Integrable Field Theories.

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1. Introduction

The G-WZW models and their gauged G/H-versions (for appropriated choice of G and H) are known to describe string theories on curved backgrounds [1, 2]. The simplest examples are the $SL(2,\mathbb{R})$-WZW model, representing string AdS_3 target space time [3] and $SL(2)/U(1)$-WZW , giving rise to 2-d black hole geometry [1, 2]. In this paper we consider specific relevant perturbations $(\hat{G}, G_0, \mu_{ab})$ [4, 5, 6] of certain physicaly interesting (gauged) WZW models

$$\mathcal{L}^{\rm pert}(\hat{G}, G_0) = \mathcal{L}^{WZW}_{G_0}(g_0) - \frac{k}{2\pi}\mu_{ab}\,{\rm Tr}\left(T_a g_0 T_b g_0^{-1}\right) \tag{1.1}$$

where $G_0 \subset \hat{G}$ is finite dimensional subgroup of the affine group $\hat{G}$ with generators T_a, $g_0 \in G_0$ and μ_{ab} are real parameters. They are expected to describe the *nonconformal* counterparts of the Maldacena string/gauge theory correspondence [7] (i.e., deformations of

AdS_d/CFT_{d-1} [8]). The first question to be addressed is about the classical (and quantum) integrability of the models (1.1), i.e., for a given affine group $\hat{G}$, how to chose $G_0 \subset \hat{G}$ and μ_{ab} such that (1.1) is exactly integrable? Another important question concerns the nonperturbative topologically stable solutions of the model (1.1) and their particle or strings interpretation.

The main purpose of the present paper is the description of the semiclassical spectrum of the finite energy topological solutions (solitons and solitonic strings) of a class of integrable perturbations that preserve (a) one local $\mathrm{U}(1)(p=1)$ or (b) two $\mathrm{U}(1)\otimes\mathrm{U}(1)(p=2)$ local symmetries. The simplest representative of type (a) integrable models (IMs) studied in this paper is given by the lagrangian [9]

$$\mathcal{L}^u_{p=1} = \frac{1}{2}\eta_{ij}\partial\varphi_i\bar{\partial}\varphi_j + \frac{n}{2(n+1)}\partial R_u\bar{\partial}R_u + \partial\chi_u\bar{\partial}\psi_u e^{\beta(R_u-\varphi_1)} - V_u \tag{1.2}$$

with potential

$$V_u = \frac{m^2}{\beta^2}\left(\sum_{i=1}^{n-1} e^{\beta(\varphi_{i-1}+\varphi_{i+1}-2\varphi_i)} + e^{\beta(\varphi_1+\varphi_{n-1})}\left(1+\beta^2\psi_u\chi_u e^{\beta(R_u-\varphi_1)}\right) - n\right) \tag{1.3}$$

where $\varphi_0 = \varphi_n = 0, i, j = 1, 2, \ldots, n-1$, $\beta^2 = -2\pi/k$ and $\eta_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$ is the A_{n-1} Killing form. The above $\mathcal{L}^u_{p=1}$ is indeed a special case of the $\mathcal{L}^{\mathrm{pert}}(\hat{G}, G_0)$ (1.1) obtained by taking $\hat{G} = A^{(1)}_n$, $G_0 = \mathrm{SL}(2)\otimes\mathrm{U}(1)^{n-1}$, $\mu_{ab} = \mu_a\bar{\mu}_b$ and

$$\begin{aligned}
\mu_a T_a &= \epsilon_+ = m\left(\sum_{i=2}^{n} E^{(0)}_{\alpha_i} + E^{(1)}_{-(\alpha_2+\cdots+\alpha_n)}\right), \\
\bar{\mu}_b T_b &= \epsilon_- = m\left(\sum_{i=2}^{n} E^{(0)}_{-\alpha_i} + E^{(-1)}_{(\alpha_2+\cdots+\alpha_n)}\right)
\end{aligned} \tag{1.4}$$

The fields $\varphi_i, R_u, \psi_u, \chi_u$ that appear in (1.2) parametrize the $g_0 \in G_0$ group element as follows

$$g_0 = \exp\left(\beta\chi_u E^{(0)}_{-\alpha_1}\right)\exp\left(\beta\lambda_1\cdot H^{(0)}R_u + \beta\sum_{i=2}^{n}\varphi_{i-1}h^{(0)}_i\right)\exp\left(\beta\psi_u E^{(0)}_{\alpha_1}\right) \tag{1.5}$$

An important property of these IMs is their invariance under local U(1) transformations ($\beta = i\beta_0$)

$$R'_u = R_u + \beta_0\left(w(z)+\bar{w}(\bar{z})\right), \qquad \psi'_u = \psi_u e^{-i\beta_0^2 w(z)}, \qquad \chi'_u = \chi_u e^{-i\beta_0^2\bar{w}(\bar{z})}, \qquad \varphi'_j = \varphi_j \tag{1.6}$$

where $w, \bar{w}$ are arbitrary chiral functions. The IM (1.2) represents A_{n-1}-abelian affine Toda theory interacting with the thermal perturbation (with $\Phi^{(j=1)}_{ab}$ [11]) of the $\mathrm{SL}(2,\mathbb{R})$-WZW model. Two particular cases should be mentioned: (1) $n=1$ (no φ_i at all) and the IM (1.2) just coincides with deformed $\mathrm{SL}(2,\mathbb{R})$-WZW model (2) for $n=2$ (one φ only) it gives rise to an integrable deformation of the $\mathrm{SL}(3,R)$-Bershadsky-Polyakov model [13, 12], i.e. certain energy perturbation of the $W^{(2)}_3$-algebra minimal models [13].

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An important feature of IM (1.2) is that for imaginary coupling $\beta = i\beta_0$ (and $n \geq 2$) its potential V_u has n-distinct zeros and as a consequence the model (1.2) admits U(1)-charged topological solitons. The main characteristic of its soliton spectrum is the presence of massive $(j_\varphi, 0)$ as well as massless $(0, j_w)$ solitons (and solitonic strings) interacting with each other (j_φ, j_w are the topological charges of fields φ_l and R_u). The massless sector is represented by $\mathcal{G}_0^0 = \mathrm{U}(1)$ chiral *conformal field theory* (CFT). In fact, together with the constant vacua solutions ($E = P = 0$, $j_\varphi = j_w = 0$)

$$\varphi_l^{\mathrm{vac}} = \frac{2\pi}{\beta_0}\frac{l}{n}N\,, \qquad \psi^{\mathrm{vac}}\chi^{\mathrm{vac}} = 0\,, \qquad R^{\mathrm{vac}} = \frac{2\pi}{\beta_0}a_R \tag{1.7}$$

($N \in Z$ and a_R is a real parameter), the equations of motion of IM (1.2) admits also massless (1-D string-like) conformal solutions with nonvanishing energy $E = \pm P \neq 0$, and $j_w \neq 0$. They have again the form (1.7), but with R^{vac} replaced by R^{CFT}, i.e.,

$$R^{\mathrm{CFT}} = \beta_0\left(w(z) + \bar{w}(\bar{z})\right) \tag{1.8}$$

The spectrum of the left-moving solitons of this chiral U(1)-CFT (with appropriate periodic b.c.'s for $w, \bar{w}$), derived in section 3.4 has the form:

$$\begin{aligned} E^{\mathrm{CFT}}_{L-\mathrm{sol}} &= \frac{4n}{(n+1)\beta_0^2}\left(s + \frac{j_w}{n}\right)^2 |a_0|\,, \\ j_w &= j_\varphi - 2j_q\,, \qquad\qquad j_\varphi, j_q = 0, \pm 1, \ldots, \pm(n-1) \bmod n \\ Q_0 &= \frac{4\pi n}{n+1}\left(s + \frac{j_w}{n}\right), \qquad s = 0, \pm 1, \ldots \end{aligned} \tag{1.9}$$

where $|a_0|$ is an arbitrary infrared mass scale and Q_0 denotes the left-U(1) charge. The conformal sector of the theory provides each one of the n-vacua states ($E^v = 0$) with a tower of conformal states (j_φ, j_w, s) with $E^{\mathrm{CFT}} \neq 0$. The *massless* solitons are topologically stable solutions that interpolate between the vacua $(0,0,0)$ and an arbitrary conformal state $(0, j_w, s)$. Together with such massless solutions, it is natural to expect the existence of two types of *massive* solitons:

1. interpolating between different vacua, called g-solitons: $(0,0,0) \to (j_\varphi, 0, 0)$
2. interpolating between vacua and arbitrary exited conformal states, called u-solitons: $(0,0,0) \to (j_\varphi, j_w, s)$

as it is shown by the diagram in figure 1.

In order to construct such soliton solutions and to derive their semi-classical spectrum, it is crucial to observe that the type (1) g-solitons in fact coincide with the U(1)-charged topological solitons of the gauged dyonic IM [9, 10] (with one global U(1) symmetry) obtained from (1.2) by axial gauging of the local U(1):

$$\mathcal{L}_g^{\mathrm{ax}}(p=1) = \frac{1}{2}\sum_{i,j=1}^{n-1}\eta_{ij}\partial\varphi_i\bar{\partial}\varphi_j + \frac{\partial\chi_g\bar{\partial}\psi_g e^{-\beta\varphi_1}}{1 + \beta^2\frac{n+1}{2n}\psi_g\chi_g e^{-\beta\varphi_1}} - V_g\,, \tag{1.10}$$

$$V_g = \frac{m^2}{\beta^2}\left(\sum_{i=1}^{n-1} e^{-\beta\eta_{ij}\varphi_j} + e^{\beta(\varphi_1+\varphi_{n-1})}\left(1 + \beta^2\psi_g\chi_g e^{-\beta\varphi_1}\right) - n\right)$$

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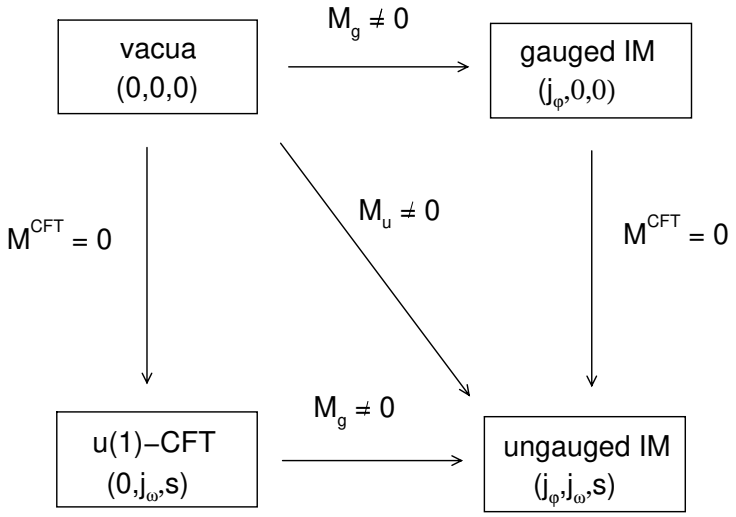

Figure 1: Relations between solitons of gauged, ungauged and CFT models.

The semiclassical spectrum of the 1-solitons of this gauged IM has been derived in our recent paper [9].

The type (2) u-solitons turns out to be the conformal dressing of the g-solitons, i.e. by performing specific U(1) gauge transformations (1.6) (with w and $\bar{w}$ given by eq. (3.40)) of the already known g-solitons [9, 10]. The spectrum of these new u-solitons can be obtained by applying the following nonconformal version of the Goddard-Kent-Olive (GKO) coset construction [14],

$$T^u_{G_0} = T^g_{G_0/G_0^0} + T^{\rm CFT}_{G_0^0}\,, \qquad E^u = E^g + E^{\rm CFT} \tag{1.11}$$

establishing the relation between the stress-tensors (and energies) of the ungauged IM (1.2), the gauged IMs (1.10) and the U(1)-CFT. The spectrum of the left-moving u-solitons ($w \neq 0$, $\bar{w} = 0$), derived in section 3.5, has the form,

$$\begin{aligned}
M_u^2 &= M_g \left(M_g + \frac{8nm_0}{(n+1)\beta_0^2} \left(s + \frac{j_w}{n} \right)^2 \right), \\
M_g &= \frac{4mn}{\beta_0^2} \left| \sin\left(\frac{4\pi j_\varphi - \beta_0^2 j_{\rm el}}{4n} \right) \right|, \qquad Q_{\rm el} = \beta_0^2 j_{\rm el}\,, \qquad j_{\rm el} = 0, \pm 1, \cdots \\
Q^u_R &= \frac{n+1}{2n} \left(Q_{\rm el} + \frac{2\pi n}{n+1} \left(s + \frac{j_w}{n} \right) \right), \qquad Q^u_\theta = \frac{\pi}{2} \left(s + \frac{j_w}{n} \right)
\end{aligned}$$

where $m_0 = |a_0| e^{b_L}$ and g-soliton velocity v_g coincides with the left rapidity b_L, i.e., $v_g = b_L$. The stability of these strong coupling particles is ensured by their nontrivial topological charges: j_φ, s, j_w.

The paper is organized as follows. In section 2 we present a brief introduction to the path integral version of the hamiltonian reduction method. The lagrangians of the different IMs (with one local U(1), two local $\mathrm{U}(1) \otimes \mathrm{U}(1)$ and one local and one global U(1)'s) considered in this paper are derived together with the proof of their integrability. Section 2.3 is devoted to the identification of the above IMs as relevant integrable deformations of

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the conformal minimal models of certain extended conformal algebras $W_{n+1}^{(n)}$ and $V_{n+1}^{(1,1)}$. The discussion of section 3 is concentrated on the symmetries (continuous U(1) and discrete $Z_n \otimes Z_2$), their currents and especially on the allowed b.c's for the fields of IMs (1.2) and (1.10). The main result of section 3.2 is the derivation of the *nonconformal* GKO coset construction. In section 3.4, we study the properties of the different U(1)-CFT solitons: left, right and left-right ones. The spectrum of the composite u-solitons is obtained in section 3.5. The spectral flow of the U(1)-charges $Q_0, \bar{Q}_0$ induced from certain topological θ-terms added to the original lagrangian (1.2) is derived in section 3.6. Section 3.7 is devoted to the nontopological solitons of the deformed $SL(2,\mathbb{R})$-WZW model. The constructions of different soliton solutions of the ungauged IM (2.14) with $U(1) \otimes U(1)$ local symmetries as well as of the so called intermediate IM are presented in section 4. Section 5 contains our conclusions and few remarks concerning the use of the charge spectrum of our soliton and string solutions in the off-critical (i.e. nonconformal) AdS_3/CFT_2 correspondence. The IMs with local $SL(2,\mathbb{R}) \otimes U(1)$ symmetry and an example of nonrelativistic IM with local U(1) symmetry are also discussed.

2. Integrable perturbations of gauged $A_n^{(1)}$-WZW model

2.1 Effective lagrangians from hamiltonian reduction

Different integrable perturbations $(\hat{G}, G_0, \mu_{ab})$ of the gauged WZW model (1.1) are known to be (at least classically) in one to one correspondence to the admissible graded structures $(\hat{\mathcal{G}}, Q, \epsilon_\pm)$ of the affine algrebra $\hat{\mathcal{G}}$

$$[Q, \mathcal{G}_k] = k\mathcal{G}_k\,, \qquad [\mathcal{G}_k, \mathcal{G}_l] \subset \mathcal{G}_{k+l}\,, \qquad k, l \in Z\,, \qquad \mathcal{G}_k \subset \hat{\mathcal{G}}\,, \qquad [Q, \epsilon_\pm] = \pm\epsilon_\pm \tag{2.1}$$

introduced by an appropriate grading operator[1]

$$Q = \tilde{h}d + \sum_{i=1}^{\text{rank }\hat{\mathcal{G}}=n} s_i \lambda_i \cdot H^{(0)}\,, \quad [d, E_\alpha^{(a)}] = aE_\alpha^{(a)}\,, \qquad a, s_i, \tilde{h} \in Z \tag{2.2}$$

such that the zero graded subalgebra $\mathcal{G}_0$, i.e., $[Q, \mathcal{G}_0] = 0$ is finite dimensional. The functional integral version of the hamiltonian reduction method [15, 16, 10] consists in considering the two-loop (i.e., affine) $\hat{G}$-WZW model [17] and imposing certain constraints on the $\hat{\mathcal{G}}$-currents $J_\alpha^{(a)} = \text{Tr}(g^{-1}\partial g E_\alpha^{(a)})$ and $\bar{J}_{-\alpha}^{(a)} = \text{Tr}(\bar{\partial} g g^{-1} E_{-\alpha}^{(a)})$ from the positive and negative grades $|q| \geq 1$ subalgebras[2] $\mathcal{H}_\pm = \oplus_{l \in Z_\pm} \hat{\mathcal{G}}_l$. More precisely, we introduce the $\mathcal{H}_\pm$-invariant gauged $\mathcal{H}_- \backslash \hat{G} / \mathcal{H}_+$-WZW model

$$\begin{aligned} S(g, A_+, \bar{A}_-) = {} & S_{\text{WZW}}^{\hat{G}}(g) - \\ & - \frac{k}{2\pi} \int d^2x\, \text{Tr}\left(A_- \left(\bar{\partial} g g^{-1} - \epsilon_+\right) + \bar{A}_+ \left(g^{-1}\partial g - \epsilon_-\right) + A_- g \bar{A}_+ g^{-1}\right) \end{aligned} \tag{2.3}$$

[1] λ_i are the fundamental weights and $\alpha_i \cdot H^{(a)}$, $E_\alpha^{(a)}$ the generators of $\hat{\mathcal{G}}$. The operator d is the derivation operator.

[2] these are the simplest grade $|q| = 1$ IMs. Imposing constraints on $J_\alpha^{(a)}$, $\bar{J}_{-\alpha}^{(a)}$ of grades $|q| \geq s$, one can construct in this way the so called higher grades IMs [18].

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where $\bar{A}_+(z,\bar{z}) \in \mathcal{H}_+$, $A_-(z,\bar{z}) \in \mathcal{H}_-$, $g(z,\bar{z}) \in \hat{G}$ and $\epsilon_\pm$ are specific constant linear combinations of grade ± 1 generators of $\hat{\mathcal{G}}$, say,

$$\epsilon_+ = \mu_a T_a = \sum_i \mu_i E^{(0)}_{\alpha_i} + \mu_0 E^{(1)}_{-\sigma}\,, \qquad \epsilon_- = \bar{\mu}_a T_a = \sum_i \bar{\mu}_i E^{(0)}_{-\alpha_i} + \bar{\mu}_0 E^{(-1)}_{\sigma}\,,$$

where $\mu_{ab} = \mu_a \bar{\mu}_b$ and σ is a fixed composite root, such that $[Q, E^{(\pm 1)}_{\mp\sigma}] = \pm E^{(\pm 1)}_{\mp\sigma}$, α_i denote the simple roots of $\hat{\mathcal{G}}$. Then the effective lagrangian (1.1) appears as a result of performing the gaussian integral on A_- and $\bar{A}_+$ in the partition function

$$\mathcal{Z} = \int DgDA_-D\bar{A}_+ \exp(-S(g, A_-, \bar{A}_+)) \sim \int Dg_0 \exp(-S_{\rm pert}(g_0)) \tag{2.4}$$

of the gauged two loop WZW model, where $g_0(z,\bar{z}) \in G_0 \subset \hat{G}$. By construction for each admissible graded structure (2.2) of $\hat{G}$ and for any choice of $\epsilon_\pm$ the corresponding $\mathcal{L}_{\rm pert}$ (1.1) represents an integrable model, as we shall demonstrate in the next subsection 2.2. Therefore the individual properties of such IMs are determined by G_0 and $\epsilon_\pm$, containing all the information about the physical fields $g_0 = g_0(\varphi_l, \psi_a, \chi_a, \ldots)$ and their nonconformal interactions. Depending on the grading operator Q the zero grade subgroup $G_0 \subset \hat{G}$ can be *abelian* $G_0 = \mathrm{U}(1)^l$, $0 \le l \le n$, or *nonabelian*, say $G_0 = \mathrm{SL}(2)^p \otimes \mathrm{U}(1)^{n-p}$. For example for $\hat{G} = A^{(1)}_n$, taking Q in the form (principal gradation) $Q = (n+1)d + \sum_{i=1}^n \lambda_i \cdot H^{(0)}$ one conclude that $G_0 = \mathrm{U}(1)^n$. We next choose the most general $\epsilon_\pm$, such that $[Q, \epsilon_\pm] = \pm\epsilon_\pm$,

$$\epsilon_\pm = \sum_{i=1}^n \mu^\pm_i E^{(0)}_{\pm\alpha_i} + \mu^\pm_0 E^{(\pm 1)}_{\mp(\alpha_1 + \cdots + \alpha_n)} \tag{2.5}$$

and observe that the corresponding $\mathcal{L}_{\rm pert}$ (1.1) takes the form of the well known abelian affine Toda model [19]. It represents an integrable perturbation of the W_{n+1}-minimal models [20] which describes marginal ($\mu^\pm_i$) and relevant ($\mu^\pm_0$) perturbations of a string on flat background with certain tachyonic potential. An interesting nonflat strings backgrounds are provided by the non-abelian (NA) affine Toda models [19], i.e., when $G_0 \subset \hat{G}$ is non-abelian. The simplest example of such model is defined by the following algebraic data:

$$\hat{G} = A^{(1)}_n\,, \qquad G_0 = \mathrm{SL}(2) \otimes \mathrm{U}(1)^{n-1}\,, \qquad Q = nd + \sum_{i=2}^n \lambda_i \cdot H^{(0)} v,$$
$$\epsilon_\pm = m \sum_{i=2}^n E^{(0)}_{\pm\alpha_i} + E^{(\pm)}_{\mp(\alpha_2 + \cdots + \alpha_n)}\,, \tag{2.6}$$

and physical fields parametrizing the zero grade group element

$$g_0 = e^{\beta\chi_u E^{(0)}_{-\alpha_1}} e^{\beta\lambda_1 \cdot H R_u + \beta \sum_{i=1}^{n-1} \varphi_i h_{i+1}} e^{\beta\psi_u E^{(0)}_{\alpha_1}}$$

Its lagrangian derived from eqs. (2.4) and (2.4) has the form (1.2) and represents an integrable perturbation of G_0-WZW model. The main new feature of this $A^{(1)}_n (p=1)$ affine NA-Toda model in comparison with the abelian one defined by eq. (2.5), is the existence of nontrivial invariant subalgebra $\mathcal{G}^0_0 \subset \mathcal{G}_0$, such that $[\mathcal{G}^0_0, \epsilon_\pm] = 0$, i.e. $\mathcal{G}^0_0 = \mathrm{U}(1) = \{\lambda_1 \cdot H^{(0)}\}$.

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It contains all the information about the continuous symmetries of the model. For the abelian affine Toda we have $\mathcal{G}_0^0 = \emptyset$, i.e., no continuous symmetries. In the NA-Toda case, (i.e., eq. (1.1) and (2.6) leading to (1.2)) one can easely verify that the lagrangian (1.1) (and therefore (1.2)) is invariant under *chiral* $\mathcal{G}_0^0 = \mathrm{U}(1)$ transformations;

$$g_0' = e^{\beta\bar{w}(\bar{z})\lambda_1\cdot H} g_0 e^{\beta w(z)\lambda_1\cdot H} \tag{2.7}$$

which is the compact form of the field transformation (1.6).

Similarly to the *conformal* (i.e. unperturbed) WZW models one can further gauge fix the above chiral $\mathcal{G}_0^0$-symmetry. The standard procedure of gauge fixing [21] consists in considering the following "improved" action by the addition of auxiliary $\mathcal{G}_0^0$-fields $A_0(z,\bar{z})$, $\bar{A}_0(z,\bar{z})$, playing the role of Lagrange multipliers:

$$\begin{aligned} S(g_0, A_0, \bar{A}_0) = {} & S_u(g_0) - \\ & - \frac{k}{2\pi}\int d^2x\, \mathrm{Tr}\left(\pm A_0\bar{\partial}g_0 g_0^{-1} + \bar{A}_0 g_0^{-1}\partial g_0 \pm A_0 g_0 \bar{A}_0 g_0^{-1} + A_0\bar{A}_0\right) \end{aligned} \tag{2.8}$$

where the signs $\pm$ takes place for axial/vector gaugings of the U(1), respectively. The new $A_0, \bar{A}_0$-dependent terms added to the action $S_u(g_0)$ are responsible for imposing the additional constraints $J_{\lambda_1\cdot H} = \mathrm{Tr}\left(\lambda_1 \cdot H g_0^{-1}\partial g_0\right) = \bar{J}_{\lambda_1\cdot H} = \mathrm{Tr}\left(\lambda_1 \cdot H \bar{\partial} g_0 g_0^{-1}\right) = 0$. They promote the chiral U(1) symmetry (2.7) to the following local U(1) symmetry $\alpha_0 = \alpha_0(z,\bar{z}) \in \mathrm{U}(1)$

$$g_0' = \alpha_0 g_0 \alpha_0'\,, \qquad \bar{A}_0' = \bar{A}_0 - \bar{\partial}\alpha_0'(\alpha_0')^{-1}\,, \qquad A_0' = A_0 - \alpha_0^{-1}\partial\alpha_0 \tag{2.9}$$

where $\alpha_0' = \alpha_0$ in the case of axial gauging and $\alpha_0' = \alpha_0^{-1}$ for the vector gauging. Again, one can take the gaussian integral on $A_0, \bar{A}_0$ as in eq. (2.4) and the result is the effective lagrangian (1.10) of the axial singular affine NA-Toda denoted as dyonic $A_n^{(1)}(p=1)$ IMs ($p = 1$ is the number of the gauged fixed U(1)'s).

According to refs. [9, 22] the CPT invariant vector gauged lagrangian has the following form:

$$\begin{aligned} \mathcal{L}_g^v(p=1) = {} & \frac{1}{2}\sum_{l=1}^{n-1}\left(\partial\phi_l\bar{\partial}(\phi_l + \phi_{l+1} + \cdots + \phi_{n-1}) + \bar{\partial}\phi_l\partial(\phi_l + \phi_{l+1} + \cdots + \phi_{n-1})\right) - \\ & - \frac{1}{2}\frac{\partial A\bar{\partial}B + \bar{\partial}A\partial B}{1-\beta^2 AB} - V_v\,, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} V_v = {} & \frac{m^2}{\beta^2}\left(Ae^{\beta(2\phi_1+\phi_2+\cdots+\phi_{n-1})} + Be^{-\beta(2\phi_{n-1}+\phi_{n-2}+\cdots+\phi_1)}\right) + \\ & + \frac{m^2}{\beta^2}\left(\sum_{k=1}^{n-1} e^{\beta(\phi_{k+1}-\phi_k)} - n\right) \end{aligned} \tag{2.11}$$

The group element $g_0^v \in \mathrm{SL}(2)\otimes\mathrm{U}(1)^{n-1}/\,\mathrm{U}(1)$ in the vector case is parametrized by the fields A, B and ϕ_l as follows

$$\begin{aligned} g_0^v &= \begin{pmatrix} d_2 & 0 \\ 0 & d_{n-1} \end{pmatrix}\,, \qquad d_{n-1} = \mathrm{diag}\left(e^{\beta\phi_1}, \ldots, e^{\beta\phi_{n-1}}\right), \\ d_2 &= \begin{pmatrix} A & u \\ \frac{AB-1}{ue^{\beta(\phi_1+\cdots+\phi_{n-1})}} & Be^{-\beta(\phi_1+\cdots+\phi_{n-1})} \end{pmatrix} \end{aligned}$$

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The nonlocal field u (i.e., the vector analog of R_g) is defined by the following first order equations [9]:

$$\partial \ln \left(u e^{\beta(\phi_1+\cdots+\phi_{n-1})} \right) = -\beta^2 \frac{(A\partial B)}{1-\beta^2 AB} V \,, \qquad \bar\partial \ln u = -\beta^2 \frac{(B\bar\partial A)}{1-\beta^2 AB} \,.$$

The main difference between "ungauged" IM $\mathcal{L}^u(p=1)$ (1.2) and its gauged, axial (1.10) and vector (2.11) versions,is that the latters are invariant only under *global* U(1) transformations, say ($\beta = i\beta_0$)

$$A' = e^{ia\beta_0^2} A \,, \qquad B' = e^{-ia\beta_0^2} B \,, \qquad \phi_l' = \phi_l - \frac{a}{n}\beta_0 \tag{2.12}$$

They contain one less field (no R_u or u) and represent more complicated target space metric $g_{AB} \sim \frac{1}{1-\beta^2 AB}$, than the one of the ungauged model (1.2). The reason we are considering them once more in the present paper is due to the crucial role they are going to play (see section 3.3) in the description of different 1-soliton solutions of the ungauged IM (1.2).

Together with the IMs (1.2) with one *local* U(1) symmetry we shall study the soliton spectrum of another class of *multicharged* $A_n^{(1)}(p=2)$ IMs [23, 24] with two local U(1)⊗U(1) symmetries and also IMs with one local and one global U(1) described by lagrangian (2.18) below. The starting point in the descrition of their effective lagrangians is the following specific *graded structure* of $\hat G = A_n^{(1)}$:

$$\begin{aligned}
Q &= (n-1)d + \sum_{i=2}^{n-1} \lambda_i \cdot H^{(0)} \,, \qquad G_0 = \mathrm{SL}(2)\otimes \mathrm{SL}(2) \otimes \mathrm{U}(1)^{n-2} \,, \quad \mathcal{G}_0^0 = \mathrm{U}(1)\otimes \mathrm{U}(1) \,, \\
\epsilon_\pm &= m \left(\sum_{i=2}^{n-1} E^{(0)}_{\pm\alpha_i} + E^{(\pm 1)}_{\mp(\alpha_2+\cdots+\alpha_{n-1})} \right) , \\
g_0 &= e^{\beta \sum_{a=1,n} \chi_a^u E^{(0)}_{-\alpha_a}} e^{\beta \sum_{a=1,n} \lambda_a \cdot H R_a^u + \beta \sum_{i=1}^{n-2} \varphi_i h_{i+1}} e^{\beta \sum_{a=1,n} \psi_a^u E^{(0)}_{\alpha_a}}
\end{aligned} \tag{2.13}$$

This algebraic data was used in deriving the lagrangian (2.4) for the gauged fixed $\mathcal{H}_\pm$ two-loop WZW model. Again, by integrating the auxiliary fields A_+, $\bar A_-$, one gets (1.1). More explicitly the ungauged multicharged $A_n^{(1)}(p=2)$ IM is given by [23]

$$\begin{aligned}
\mathcal{L}^u_{p=2} &= \frac{1}{2} \sum_{i=1}^{n-2} \eta_{ij} \partial\varphi_i \bar\partial \varphi_j + \partial\chi_1^u \bar\partial \psi_1^u e^{\beta(R_1^u - \varphi_1)} + \partial \chi_n^u \bar\partial \psi_n^u e^{\beta(R_n^u - \varphi_{n-2})} + \\
&\quad + \frac{1}{2(n+1)} \left(n \partial R_1^u \bar\partial R_1^u + n \partial R_n^u \bar\partial R_n^u + \partial R_1^u \bar\partial R_n^u + \partial R_n^u \bar\partial R_1^u \right) - V_u^{p=2}
\end{aligned} \tag{2.14}$$

with potential

$$\begin{aligned}
V_u^{p=2} &= \frac{m^2}{\beta^2} \left(\sum_{i=1}^{n-2} e^{-\beta \eta_{ij} \varphi_j} + e^{\beta(\varphi_1 + \varphi_{n-2})} \left(1 + \beta^2 \psi_1^u \chi_1^u e^{\beta(R_1^u - \varphi_1)} \right) \times \right. \\
&\qquad \left. \times \left(1 + \beta^2 \psi_n^u \chi_n^u e^{\beta(R_n^u - \varphi_{n-2})} \right) - n + 1 \right) ,
\end{aligned} \tag{2.15}$$

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where $\varphi_0 = \varphi_{n-1} = 0$. For the specific choice of constant grade ± 1 elements $\epsilon_\pm$ (2.13), the form of the invariant subalgebra $\mathcal{G}_0^0 = \{\lambda_1 \cdot H, \lambda_n \cdot H\}$ is an indication that the IM (2.14) is invariant under the following chiral $\mathrm{U}(1) \otimes \mathrm{U}(1)$ transformations

$$g_0' = e^{\beta \sum_a \lambda_a \cdot H \bar{w}_a(\bar{z})} g_0 e^{\beta \sum_a \lambda_a \cdot H w_a(z)} \tag{2.16}$$

with $\bar{\partial} w_a = \partial \bar{w}_a = 0$. The field transformation encoded in eq. (2.16) have the form

$$(R_a^u)' = R_a^u + \beta_0(w_a + \bar{w}_a)\,, \qquad \varphi_l' = \varphi_l\,, \qquad (\psi_a^u)' = e^{-i\beta_0^2 w_a}\psi_a^u\,, \qquad (\chi_a^u)' = e^{-i\beta_0^2 \bar{w}_a}\chi_a^u \tag{2.17}$$

By axial or vector gauge fixing one of the local symmetries (say, the one generated by $(\lambda_1 + \lambda_n) \cdot H$) one can derive an interesting "intermediate" IM with one local and one global U(1) symmetries. We take eq. (2.9) with $g_0 \in G_0 = \mathrm{SL}(2) \otimes \mathrm{SL}(2) \otimes \mathrm{U}(1)^{n-2}$ and $A_0 = a_0(z,\bar{z})(\lambda_1 + \lambda_n) \cdot H$, $\bar{A}_0 = \bar{a}_0(z,\bar{z})(\lambda_1 + \lambda_n) \cdot H$, ($a_0, \bar{a}_0$ are arbitrary functions) and by performing the gaussian integration over a_0, $\bar{a}_0$ we obtain the effective lagrangian for the intermediate axial IM [23]

$$\begin{aligned}
\mathcal{L}_{p=2}^{\mathrm{interm}} = {} & \frac{n-1}{n+1}\partial\bar{R}\bar{\partial}\bar{R} + \frac{1}{2}\sum_{i=1}^{n-2}\eta_{ij}\partial\varphi_i\bar{\partial}\varphi_j + \\
& + \frac{1}{\Delta_0}\Bigg(\left(1 + \frac{\beta^2}{4}\bar{\psi}_n\bar{\chi}_n e^{-\beta(\varphi_{n-2}+\bar{R})}\right)\bar{\partial}\bar{\psi}_1\partial\bar{\chi}_1 e^{\beta(\bar{R}-\varphi_1)} + \\
& \quad + \left(1 + \frac{\beta^2}{4}\bar{\psi}_1\bar{\chi}_1 e^{-\beta(\varphi_1+\bar{R})}\right)\bar{\partial}\bar{\psi}_n\partial\bar{\chi}_n e^{-\beta(\bar{R}+\varphi_{n-2})} - \\
& \quad - \frac{\beta^2}{4}\left(\bar{\psi}_n\bar{\chi}_1\bar{\partial}\bar{\psi}_1\partial\bar{\chi}_n + \bar{\psi}_1\bar{\chi}_n\bar{\partial}\bar{\psi}_n\partial\bar{\chi}_1\right)e^{-\beta(\varphi_1+\varphi_{n-2})}\Bigg) - V_{\mathrm{interm}}
\end{aligned} \tag{2.18}$$

with potential

$$\begin{aligned}
V_{\mathrm{interm}} = {} & \\
& \frac{m^2}{\beta^2}\Bigg(\sum_{i=1}^{n-2} e^{-\beta\eta_{ij}\varphi_j} + \\
& \quad + e^{\beta(\varphi_1+\varphi_{n-2})}\left(1 + \beta^2\bar{\psi}_1\bar{\chi}_1 e^{\beta(\bar{R}-\varphi_1)}\right)\left(1 + \beta^2\bar{\psi}_n\bar{\chi}_n e^{-\beta(\bar{R}+\varphi_{n-2})}\right) - n + 1\Bigg)
\end{aligned}$$

and the denominator Δ_0 is quadratic in $\bar{\psi}_a, \bar{\chi}_a$:

$$\Delta_0 = 1 + \frac{\beta_0^2}{4}\left(\bar{\psi}_1\bar{\chi}_1 e^{\beta(\bar{R}-\varphi_1)} + \bar{\psi}_n\bar{\chi}_n e^{-\beta(\bar{R}+\varphi_{n-2})}\right)$$

The relations between the fields of the ungauged (2.14) and the above axial gauged IM (2.18) has the form

$$\bar{\psi}_a = \psi_a^u e^{\frac{\beta}{4}(R_1^u + R_n^u)}\,, \qquad \bar{\chi}_a = \chi_a^u e^{\frac{\beta}{4}(R_1^u + R_n^u)}\,, \qquad \bar{R} = \frac{1}{2}(R_1^u - R_n^u) \tag{2.19}$$

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as one can see by comparing the group elements $g_0 \in G_0 = \mathrm{SL}(2) \otimes \mathrm{SL}(2) \otimes \mathrm{U}(1)^{n-2}$ from eq. (2.13) with $g_0^f \in G_0/\mathrm{U}(1)$ (for axial gauging), i.e.,

$$g_0 = e^{\frac{\beta}{2}(\lambda_n \cdot HR_1^u + \lambda_1 \cdot HR_n^u)} g_0^{f,\mathrm{interm}} e^{\frac{\beta}{2}(\lambda_n \cdot HR_1^u + \lambda_1 \cdot HR_n^u)} \tag{2.20}$$

where $g_0^{f,\mathrm{interm}} = e^{\beta \sum_{a=1,n} \bar{\chi}_a E^{(0)}_{-\alpha_a}} e^{\beta(\lambda_1 - \lambda_n)\cdot H\bar{R} + \beta \sum_{i=1}^{n-2} \varphi_i h_{i+1}} e^{\beta \sum_{a=1,n} \bar{\psi}_a E^{(0)}_{\alpha_a}}$. Note that the $p = 2$ lagrangian (2.18) is quite similar to (1.10) of the $p = 1$ gauged IM. Both are invariant under one global U(1) symmetry. The denominators of both are quadratic in ψ_a, χ_a, but (2.18) involves an extra pair ψ_n^u, χ_n^u of charged fields and it is invariant under the following chiral U(1) transformation:

$$\begin{aligned} &\bar{\psi}_1' = e^{-i\beta_0^2 w}\psi_1\,, \qquad \bar{\chi}_1' = e^{-i\beta_0^2 \bar{w}}\chi_1\,, \qquad \bar{\psi}_n' = e^{i\beta_0^2 w}\psi_n\,, \qquad \bar{\chi}_n' = e^{i\beta_0^2 \bar{w}}\chi_n\,, \\ &\bar{R}' = \bar{R} + \beta_0(w + \bar{w})\,, \qquad \varphi_l' = \varphi_l\,, \end{aligned} \tag{2.21}$$

The description of the 1-soliton spectrum of this IM (2.18) is one of the main purposes of the present paper. As we shall show in the next section 4.3, the structure of its soliton solutions is quite similar to the ones of $p = 1$ ungauged IM (1.2). Due to the common local U(1) symmetry, they share the same U(1) CFT, but their generic M_u-solitons are conformal dressing of different "gauged" M_g-solitons. In the case of IM (2.18) one need to know the multicharged solitons of the corresponding completely gauged IM, i.e. the IM obtained by (axial) gauge fixing of both local $\mathrm{U}(1) \otimes \mathrm{U}(1)$ symmetries of the ungauged IM (2.14). The procedure of gauge fixing is the same as in the case of intermediate IM (2.18), but now the auxiliary fields $A_0, \bar{A}_0 \in \mathcal{G}_0^0$ are different, namely,

$$A_0 = a_{01}\lambda_1 \cdot H + a_{0n}\lambda_n \cdot H\,, \qquad \bar{A}_0 = \bar{a}_{01}\lambda_1 \cdot H + \bar{a}_{0n}\lambda_n \cdot H$$

The result of the matrix gaussian integration (over a_{0a}, $\bar{a}_{0a}$, $a = 1, n$) is the following effective lagrangian of the completely axialy gauged IM [23]

$$\begin{aligned} \mathcal{L}_n^{p=2} = &\frac{1}{2}\sum_{i=1}^{n-2} \eta_{ij}\partial\varphi_i\bar{\partial}\varphi_j + \\ &+ \frac{1}{\Delta}\Bigg(\left(1 + \beta^2\frac{n}{2(n-1)}\psi_n\chi_n e^{-\beta\varphi_{n-2}}\right)\bar{\partial}\psi_1\partial\chi_1 e^{-\beta\varphi_1} + \\ &\quad + \left(1 + \beta^2\frac{n}{2(n-1)}\psi_1\chi_1 e^{-\beta\varphi_1}\right)\bar{\partial}\psi_n\partial\chi_n e^{-\beta\varphi_{n-2}} + \\ &\quad + \frac{\beta^2}{2(n-1)}\left(\chi_1\psi_n\bar{\partial}\psi_1\partial\chi_n + \chi_n\psi_1\bar{\partial}\psi_n\partial\chi_1\right)e^{-\beta(\varphi_1+\varphi_{n-2})}\Bigg) - V_n^{p=2} \end{aligned} \tag{2.22}$$

with potential

$$\begin{aligned} V_n^{p=2} = \frac{\mu^2}{\beta^2}\Bigg(&\sum_{i=1}^{n-2} e^{-\beta\eta_{ij}\varphi_j} + \\ &+ e^{\beta(\varphi_1+\varphi_{n-2})}\left(1 + \beta^2\psi_n\chi_n e^{-\beta\varphi_{n-2}}\right)\left(1 + \beta^2\psi_1\chi_1 e^{-\beta\varphi_1}\right) - n + 1\Bigg) \end{aligned} \tag{2.23}$$

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where $\varphi_0 = \varphi_{n-1} = 0$ and

$$\Delta = 1 + \frac{\beta^2 n}{2(n-1)} \left(\psi_1 \chi_1 e^{-\beta\varphi_1} + \psi_n \chi_n e^{-\beta\varphi_{n-2}} \right) + \frac{\beta^4 (n+1)}{4(n-1)} \psi_1 \chi_1 \psi_n \chi_n e^{-\beta(\varphi_1 + \varphi_{n-2})} \,.$$

The fields ψ_a, χ_a are related to the charged fields ψ_a^u, χ_a^u of the ungauged IM as follows:

$$g_0 = e^{-\frac{1}{2}\lambda_a \cdot H R_a} g_0^{f,\text{axial}} e^{-\frac{1}{2}\lambda_a \cdot H R_a} \tag{2.24}$$

i.e., $\psi_a = \psi_a^u e^{\frac{1}{2}R_a}$, $\chi_a = \chi_a^u e^{\frac{1}{2}R_a}$. The IM (2.22) is invariant under global U(1) $\otimes$ U(1) transformations only ($\epsilon_a = \text{const}$),

$$\psi_a' = \psi_a e^{i\beta^2 \epsilon_a} \,, \qquad \chi_a' = \chi_a^u e^{-i\beta^2 \epsilon_a} \,, \qquad \varphi_l' = \varphi_l \tag{2.25}$$

Its charged (Q_1, Q_n) 1-solitons $M_g(p = 2)$ [23] appears as the basic ingredient in the construction of the 1-solitons of both the ungauged $p = 2$ IM (2.14) and the intermediate IM (2.18).

2.2 Zero curvarure representation and classical integrability

The proof of the (classical) integrability of the 2-d models (1.2), (1.10), (2.14), (2.18) and (2.22) introduced in section 2.1, is based on the following two basic ingredients [25]

- zero curvature representation

$$\partial \bar{\mathcal{A}} - \bar{\partial} \mathcal{A} - [\mathcal{A}, \bar{\mathcal{A}}] = 0, \qquad \mathcal{A}, \bar{\mathcal{A}} \in \oplus_{i=0,\pm 1} \mathcal{G}_i \tag{2.26}$$

  of their equation of motion.

- fundamental Poisson bracket (FPR) relation

$$\left\{ \mathcal{A}_x(x_1, t) \overset{\otimes}{,} \mathcal{A}_x(x_2, t) \right\}_{PB} = [r_{cl},\ \mathcal{A}_x(x_1, t) \otimes I + I \otimes \mathcal{A}_x(x_2, t)]\, \delta(x_1 - x_2) \,, \tag{2.27}$$

  where r_{cl} denotes the classical r-matrix and $\mathcal{A}_x = \frac{1}{2}(\mathcal{A} - \bar{\mathcal{A}})$.

The Leznov-Saveliev matrix form [19] of the equations of motion derived from the corresponding lagrangians (1.10), (2.14), (2.18) and (2.22) is

$$\bar{\partial} \left(g_0^{-1} \partial g_0 \right) + \left[\epsilon_-, g_0^{-1} \epsilon_+ g_0 \right] = 0 \,, \qquad \partial(\bar{\partial} g_0 g_0^{-1}) - \left[\epsilon_+, g_0 \epsilon_- g_0^{-1} \right] = 0 \,, \tag{2.28}$$

with $g_0(\psi_a, \chi_a, \varphi_l, R_a) \in G_0$ given by eqs. (2.6), (2.13) and (2.20). It allows us to deduce the explicit form of the 2-d pure gauge potentials, namely,

$$\mathcal{A} = -g_0 \epsilon_- g_0^{-1} \,, \qquad \bar{\mathcal{A}} = \epsilon_+ + \bar{\partial} g_0 g_0^{-1} \tag{2.29}$$

They indeed satisfy eqs. (2.26) if g_0 is a solution of eqs. (2.28). Their explicit form in terms of the fields φ_l, ψ_a, χ_a, R_a, etc. is given in refs. [9, 10, 23, 24].

The derivation of r_{cl} and the proof of eq. (2.27) is based on the explicit realization of $(\mathcal{A}_x)_{ij}$ in terms of fields and their canonical momenta. We next calculate the equal time matrix PBs:

$$\{ \mathcal{A}_x(x_1, t) \overset{\otimes}{,} \mathcal{A}_x(x_2, t) \}_{ij;kl} = \{ (\mathcal{A}_x)_{ik}, (\mathcal{A}_x)_{jl} \}$$

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making use of the basic canonical PBs, say $\{\phi_j(x,t), \Pi_{\phi_k}(y,t)\} = \delta_{jk}\delta(x-y)$. The result for the IM (1.2) has the form [26]

$$r_{cl} = \beta^2 \left(C^+ - C^-\right) \tag{2.30}$$

where

$$C^+ = \sum_{m=1}^{\infty} \left(\sum_{a,b=1}^{n} \left(K^{-1}\right)_{ab} h_a^{(m)} \otimes h_b^{(-m)} + \sum_{\alpha>0} \left(E_\alpha^{(m)} \otimes E_{-\alpha}^{(-m)} + E_{-\alpha}^{(m)} \otimes E_\alpha^{(-m)} \right) \right) + \\ + \frac{1}{2} \sum_{\alpha>0} E_\alpha^{(0)} \otimes E_{-\alpha}^{(0)}\,, \tag{2.31}$$

and $C^- = \sigma(C^+)$, $\sigma(A \otimes B) = B \otimes A$; K_{ab}^{-1} denotes the inverse of the Cartan matrix for $\mathcal{G}A_n$. It turns out that r_{cl} has an universal form (2.30), (2.31) for each given algebra, say, $A_n^{(1)}$ and does not depend on the choice of the specific graded structure and of $\epsilon_\pm$, i.e., the abelian and nonabelian affine A_n-Toda model provide different representations of the same FPR algebra (2.27).[3]

The complete proof of the integrability of the considered models require two more steps. The first is the construction of the infinite set of conserved charges. Their existence and explicit form is well known consequence of the zero curvature representation (2.26) [25], namely

$$\begin{aligned} \mathrm{Tr}(T(\tau)^m) &= P_m(\tau)\,, \qquad \partial_\tau P_m = 0 \qquad m = 0, \pm 1, \dots \\ T(\tau) &= \lim_{L\to\infty} P \exp \int_{-L}^{L} \mathcal{A}_x(\tau, x) dx \end{aligned} \tag{2.32}$$

i.e., $P_m(0)$ are the conserved charges we seek. The last step is to prove that these conserved charges are in involution, i.e.

$$\{P_{m_1}, P_{m_2}\}_{PB} = 0$$

which is ensured by the specific form of the FPR (2.27) and the explicit form of the charges P_m (2.32).

2.3 Identification of dyonic IMs as perturbed CFTs

The hamiltonian reduction [15] and the integrable relevant perturbations of certain conformal minimal models [4, 5, 6], are known to be two equivalent methods for constructing and solving a large class of IMs. As we have shown in section 2.1, the first method is based on an appropriate graded structure $(Q, G_0, \epsilon_\pm)$ of the defining affine algebra $\hat{\mathcal{G}}$ (say, $A_n^{(1)}$) and each choice of $\mathcal{G}, Q$ and $\epsilon_\pm$ determines one IMs. The second method introduced by Zamolodchikov [4, 6], consists in considering certain quantum CFT (i.e. conformal minimal model of one of the extended Virasoro algebras: conformal current algebra, W_n,

[3] It is not however true for the quantum R-matrices, which acquire specific form for each fixed graded structure and $\epsilon_\pm$.

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Z_n-parafermionic algebra, [27, 28], etc.) and adding to its lagrangian a linear combination $\sum_{a=1}^{l} g_a V_a \bar{V}_a$ of certain *relevant* vertex operators of dimension $\Delta_a + \bar{\Delta}_a < 2$ representing highest weight representations of the corresponding extended conformal algebra, i.e.,

$$\mathcal{L}_{\rm pert} = \mathcal{L}_{\rm conf} + \sum_{a=1}^{l} g_a V_a(z) \bar{V}_a(\bar{z})$$

The relation between these two methods has been demonstrated on the examples of sine-Gordon (SG) model as $\phi_{1,2}\bar{\phi}_{1,2}$ perturbation of the Virasoro minimal models [30], abelian affine Toda as W_{n+1} minimal model perturbations [20], the Lund-Regge model as $\phi^{(2)}_{0,0}$ perturbation of the Z_N parafermions [36, 32], etc. (see ref. [33] for review). The problem we address in this section is to recognize the IMs derived in section2.1 as perturbed CFTs. Therefore we have to answer the following two questions:

- which are the extended conformal algebras behind the conformal limits $V = V_{\rm conf} + V_{\rm pert} \to V_{\rm conf}$ of the dyonic IMs (1.2), (1.10), (2.14), (2.18) and (2.22)?
- how to identify the nonconformal part of their potentials $V_{\rm pert}$, say for the IM (1.10)

$$V^{(g)}_{\rm pert} = e^{\beta(\varphi_1+\varphi_{n-1})}\left(1 + \beta^2 \psi_g \chi_g e^{-\beta\varphi_1}\right) \tag{2.33}$$

  with certain linear combinations of vertex operators of the underlying conformal models (of the V_{n+1}-algebra [16, 35, 34] for the gauged IM (1.10))?

One should consider separately the U(1)-IMs (1.2) (and (1.10)) with $n = 1$ and $n \geq 2$, due to the fact that they are based on different type of extended conformal algebras, namely:

- the $n = 1$ ungauged IM (1.2) is governed by the chiral $\mathrm{SL}(2,\mathbb{R})$ conformal current algebra and its gauged version $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ - by the parafermionic algebra [27].
- the $n \geq 2$ ungauged IMs (1.2) appear to be certain perturbations of the minimal models of the $W^{(n)}_{n+1}$-Bershadsky-Polyakov algebra [13, 12]. The conformal limit of the gauged IMs (1.10) with $n \geq 2$ is characterized by the nonlocal $V^{(1)}_{n+1}$-algebra of mixed PF-W_n-type [16, 35, 34].

Similar separation takes place in the case of U(1)⊗U(1) multicharged IMs (2.14), (2.18) and (2.22). The simplest case $n = 2$ represents relevant perturbations of the $\mathrm{SL}(2,\mathbb{R}) \otimes \mathrm{SL}(2,\mathbb{R})$ WZW models, while the $n \geq 3$ case is related to specific quadratic (non-Lie) $W^{(n,2)}_{n+1}$-algebra spanned by two sets $J^{\pm}_{\alpha_a}, a = 1, n$ of spin $s = n/2$ currents, two spin $s = 1$ currents $J_{\lambda_a \cdot H}$ and $n-2$ currents T_s of spin $s = 2, 3, \ldots, n-1$. Its algebraic structure (i.e. the OPE's of these currents) is quite similar to the one of $W^{(n)}_{n+1}$ algebra, but including an extra set of currents $J^{\pm}_{\alpha_n}$, $J_{\lambda_n \cdot H}$.

We first consider few different integrable perturbations of the SL(2,R)-WZW model [36, 37, 38]. For $n = 1$ the lagrangian (1.2) takes the form

$$\mathcal{L}^{p=1}_u(n=1) = \partial R \bar{\partial} R + \partial\chi_u \bar{\partial}\psi_u e^{2\beta R} - V_u\,, \qquad V_u = m^2 \chi_u \psi_u e^{2\beta R}\,, \qquad R = \frac{1}{2} R_u \tag{2.34}$$

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Its conformal limit $V_u \to 0$, (i.e. $m^2 \to 0$) coincides with the $SL(2,\mathbb{R})$-WZW model. We next remember that the vertex operators, representing the primary fields of the discrete series of highest weight representations [11, 39, 40] are given by

$$V^j_{m,\bar{m}}(z,\bar{z}) = \psi^{j+m}\chi^{j+\bar{m}}e^{2\beta jR} \equiv \Phi^m_j(z)\bar{\Phi}^{\bar{m}}_j(\bar{z})\,, \qquad m,\bar{m} = -j,-j+1,\ldots,j \tag{2.35}$$

of conformal dimensions $\Delta_\Phi = \bar{\Delta}_{\bar{\Phi}} = \frac{j(j+1)}{k-2}$, i.e. $\Delta_V = 2\Delta_\Phi$. Therefore the perturbation of $SL(2,\mathbb{R})$ WZW model by $V^{(1)}_{0,0}$, can be represented by $\mathcal{L}^{p=1}_u (n=1)$ of eq. (2.34). Since $V_u = \mathrm{Tr}\,\epsilon_+ g_0 \epsilon_- g_0^{-1} - \mathrm{Tr}\,\epsilon_+\epsilon_- = V^{(1)}_{0,0}$ for $\epsilon_\pm = mh^{(\pm 1)}$ of grade ± 1 with respect to the homogeneous grading $Q = d$ and $g_0 \in SL(2,\mathbb{R})$ we conclude that the IM $(n=1)$ defined by the above grading structure $Q, \epsilon_\pm$ of $\hat{G}_0 = SL(2,\mathbb{R})$ is identical with the $V^{(1)}_{0,0}$ perturbation of the $SL(2,\mathbb{R})$ WZW model. Note that in the classical limit $k \to \infty$ and $\Delta_V \to 0$ and therefore the constant m has dimension 1 in mass units. If we take the most general grade ± 1 elements

$$\epsilon_\pm = m\left(a^{(\pm)}_1 h^{(\pm 1)} + a^{(\pm)}_2 E^{(\pm 1)}_\alpha + a^{(\pm)}_3 E^{(\pm 1)}_{-\alpha}\right) \tag{2.36}$$

instead of $\epsilon_\pm \sim h^{(\pm 1)}$, we get a six parameter family of integrable perturbations of the $SL(2,\mathbb{R})$ WZW model

$$\begin{aligned} V_u = &\left(\frac{1}{2}a^+_1 + a^+_3\psi\right)\left(\frac{1}{2}a^-_1 + a^-_2\chi\right) + a^-_2 a^+_3 e^{-2\beta R} + \\ &+ \left(a^+_1\psi + a^+_3\psi^2 - a^+_2\right)\left(a^-_1\chi + a^-_2\chi^2 - a^-_3\right)e^{2\beta R} \end{aligned} \tag{2.37}$$

considered in ref. [37]. For arbitrary $a^\pm_i$ both $SL(2,\mathbb{R})_{\text{left}}$ and $SL(2,\mathbb{R})_{\text{right}}$ are broken. Moreover, when $a^+_i = a^-_i = a_i$, we have

$$\left[a_1 h^{(0)} + a_2 E^{(0)}_\alpha + a_3 E^{(0)}_{-\alpha}, \epsilon_\pm\right] = 0$$

and therefore such perturbations has chiral $U(1)_{\text{left}} \otimes U(1)_{\text{right}}$ symmetries. Particular examples of chiral integrable perturbations by

$$V^\pm_u = \Phi^{\pm j}_j(z)\,, \qquad V^{(0)}_u = \Phi^0_j(z)$$

have been introduced and studied in ref. [38]. It is important to mention that exhausting the possible gradings $(Q = d)$ and $\epsilon_\pm$'s (see eq. (2.36)) one can classify all the integrable perturbations of a given WZW model, i.e. listing the admissible graded structures $(\hat{G}, Q, \epsilon_\pm)$ one separates few integrable linear combinations of the vertices $V^j_{m,\bar{m}}$ among the large set of combinations with $j \le k/2$.

In the cases when the perturbation preserves the chiral U(1) symmetry (i.e. $a^+_i = a^-_i$) one can further gauge fix this symmetry as it was explained in section 2.1 (see eq. (2.9)). The corresponding $n = 1$ gauged IMs (1.10) (or more general for $\epsilon_\pm$ given by eq. (2.36)) give rise to different integrable perturbations of the gauged $SL(2,\mathbb{R})/U(1)$-WZW (i.e. noncompact parafermions [31]) and $SU(2)/U(1)$- (i.e. compact parafermions) studied in ref. [32, 36].

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The conformal limits of IMs (1.2) with $n \geq 2$ represent certain conformal A_n-non-abelian Toda models introduced in ref. [16, 35] (see ref. [16, section 2 and 8]). They can be defined as conformal gauged $G_0 = H_- \backslash A_n / H_+$-WZW model based on the finite dimensional Lie algebra A_n, with $H_\pm \in A_n$ being the of positive and negative graded nilpotent subalgebras according to the following grading operator

$$Q^{\rm conf} = \sum_{i=2}^{n} \lambda_i \cdot H^{(0)}\,, \qquad \epsilon_\pm^{\rm conf} = \sum_{i=2}^{n} E^{(0)}_{\pm\alpha_i}\,, \tag{2.38}$$

Similar to the conformal abelian Toda theory, the nontrivial conformal part $V_{\rm conf} = {\rm Tr}(\epsilon_+^{\rm conf} g_0 \epsilon_-^{\rm conf} g_o^{-1})$ of potential $V_u (n \geq 2)$ is originated from the specific set of constraints on the A_n-WZW currents:

$$J_{-\alpha_i} = \bar{J}_{\alpha_i} = 1\,, \qquad i = 2, \ldots n\,, \qquad J_{-[\alpha]} = \bar{J}_{[\alpha]} = 0, [\alpha] = \text{non simple root} \tag{2.39}$$

encoded in $\epsilon_\pm^{\rm conf}$. The vertices $V_j = e^{-\beta \sum \eta_{ij} \varphi_i}$ of dimension (1,1) represent the screening operators of the $W^{(n)}_{n+1}$-algebra. As it is well known these constraints reduce the original A_n-chiral conformal current algebra to specific higher spin quadratic algebra of $W^{(n)}_{n+1}$-type studied by Polyakov [12] and Bershadsky [13]. For example the $n = 2$ Bershadsky-Polyakov (BP) algebra $W_3^{(2)}$ is generated by four chiral currents $T_W, G^\pm$ and J of spins $s = 2, 3/2, 3/2, 1$. It has the following OPE form [13]

$$\begin{aligned}
J(z_1)J(z_2) &= \frac{2k+3}{3z_{12}^2} + O(z_{12})\,, \qquad G^\pm(z_1)G^\pm(z_2) = O(z_{12}) \\
J(z_1)G^\pm(z_2) &= \pm\frac{1}{z_{12}} G^\pm(z_2) + O(z_{12}) \\
T_W(z_1)T_W(z_2) &= \frac{c_W}{2z_{12}^4} + \frac{2}{z_{12}^2} T_W(z_2) + \frac{1}{z_{12}} \partial T_W(z_2) + O(z_{12}) \\
T_W(z_1)G^\pm(z_2) &= \frac{3}{2z_{12}^2} G^\pm(z_2) + \frac{1}{z_{12}} \partial G^\pm(z_2) + O(z_{12}) \\
T_W(z_1)J(z_2) &= \frac{1}{z_{12}^2} J(z_2) + \frac{1}{z_{12}} \partial J(z_2) + O(z_{12}) \\
G^+(z_1)G^-(z_2) &= (k+1)\frac{(2k+3)}{z_{12}^3} + 3\frac{(k+1)}{z_{12}^2} J(z_2) + \\
&\quad + \frac{1}{z_{12}} \left(3 : J^2 : -(k+3) T(z_2) + \frac{3}{2}(k+1)\partial J \right) + O(z_{12})
\end{aligned} \tag{2.40}$$

where $c_W = \frac{8k}{k+3} - 6k - 1$ is its central charge. In fact the original constraints of the $n = 2$ BP model

$$J_{-\alpha_2} = \bar{J}_{\alpha_2} = 0\,, \qquad J_{-\alpha_1-\alpha_2} = \bar{J}_{\alpha_1+\alpha_2} = 1$$

are the image of the (2.39) under the action of particular A_2-Weyl reflection w_{α_1}, i.e.,

$$w_{\alpha_1}(\alpha) = \alpha - (\alpha \cdot \alpha_1)\alpha_1\,, \qquad w_{\alpha_1}^2 = 1\,.$$

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as it has been shown in ref. [16, section 8]. According to the analysis of the h.w. representations of the $W_3^{(2)}$ algebra (2.40) one can construct a specific class of degenerate vertex operators, that realize in the $\psi_u, \chi_u, R_u, \varphi$ variables of our model (1.2) acquire the form:

$$V_{m,\bar{m}}^{j,\alpha_{r,s},\beta_{r,s}}(z,\bar{z}) = \psi_u^{j+m}\chi_u^{j+\bar{m}}e^{\beta(\beta_{r,s}R_u+\alpha_{r,s}\varphi)} \tag{2.41}$$

where $\alpha_{r,s}$ and $\beta_{r,s}$ are certain charges defined in ref. [16, section 9] (see also [13]) For $j=1$, $m=\bar{m}=0$, $\alpha_{r,s}=\beta_{r,s}=1$ and for $j=m=\bar{m}=\beta_{r,s}=0$, $\alpha_{r,s}=2$ one recognize the two vertex operators that form the nonconformal part (i.e., the integrable perturbation) $V_{\rm pert}$ of V_u, i.e.,

$$V_{\rm pert} = e^{2\beta\varphi} + \beta^2\psi_u\chi_u e^{\beta(R+\varphi)} = V_{0,0}^{0,2,0} + \beta^2 V_{0,0}^{1,1,1}$$

Similar identifications takes place for the gauged version (1.10) of the $n=2$ IM (1.2). In this case the role of the $W_3^{(2)}$ algebra and its vertex operators (2.41) is played by the $V_3^{(1,1)}$ algebra of mixed PF-W_2-type [16, 34] (spanned by the strees tensor T_v and two PF currents $V^{\pm}$ of spin $s^{\pm}=\frac{3}{2}(1-\frac{1}{2k+3})$) and the corresponding h.w. vertex operators. The conformal extended algebras $V_{n+1}^{(1,1)}$ of the classical symmetries of the conformal limits $V_{\rm pert}\to 0$ for generic $n\geq 2$ gauged IMs (1.10) have been constructed in ref. [16]. Their quantum versions, the h.w. representations and the vertex operators are known, however only in the particular case of $n=2$, (i.e. $V_3^{(1,1)}$). Therefore, the problem of identification of $n\geq 3$ gauged (and ungauged) dyonic IMs as perturbed CFT m.m's of the $V_{n+1}^{(1,1)}$ (and $W_{n+1}^{(n)}$) algebras remains open.

3. Dyonic IMs with U(1) local symmetry

3.1 Gauged vs. ungauged IMs: conserved charges relations

The invariance of the ungauged IM lagrangian (1.2) under local U(1) transformations (1.6) (see eq. (2.7) for their matrix form) gives rise to the following chiral U(1) conserved currents:

$$\begin{aligned} J(z) &= {\rm Tr}\left(g_0^{-1}\partial g_0\lambda_1\cdot H^{(0)}\right) = \frac{2n}{n+1}\beta_0\partial R_u - 2i\beta_0^2\psi_u\partial\chi_u e^{i\beta_0(R_u-\varphi_1)}\,, \\ \bar{J}(z) &= {\rm Tr}\left(\bar{\partial}g_0 g_0^{-1}\lambda_1\cdot H^{(0)}\right) = \frac{2n}{n+1}\beta_0\bar{\partial}R_u - 2i\beta_0^2\chi_u\bar{\partial}\psi_u e^{i\beta_0(R_u-\varphi_1)} \end{aligned} \tag{3.1}$$

i.e., $\bar{\partial}J=\partial\bar{J}=0$ as one can verify by taking traces of eqs. (2.28) with $\lambda_1\cdot H^{(0)}$ (remembering that $[\lambda_1\cdot H^{(0)},\epsilon_{\pm}]=0$). Hence the solutions of eqs. (2.28) are characterised by two infinite sets of conserved charges ($m\in Z$),

$$Q_{m+1} = \oint J(z)z^{-m-1}dz\,, \qquad \bar{Q}_{m+1} = \oint \bar{J}(\bar{z})\bar{z}^{-m-1}d\bar{z} \tag{3.2}$$

Since the conservation of the chiral U(1) currents J and $\bar{J}$ is a consequence of the conservation of both vector $J_\mu^{\rm vec}=J_\mu$ and axial $J_\mu^{\rm axial}=\epsilon_{\mu\nu}J^\nu$ currents (i.e., $\partial^\mu J_\mu^{\rm vec,axial}=0$ and therefore $J_\mu=\partial_\mu\Phi$, $\Phi=\Phi(z)+\bar{\Phi}(\bar{z})$) we realize that

$$J = J_0 + J_1 = \partial\Phi(z)\,, \qquad \bar{J} = J_0 - J_1 = \bar{\partial}\bar{\Phi}(\bar{z}) \tag{3.3}$$

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It implies the following relations between axial and vector U(1)-charges $Q_{\rm vec}$, $Q_{\rm axial}$ with the J, $\bar J$-zero modes Q_0, $\bar Q_0$:

$$\begin{aligned} Q_{\rm vec} &= \int_{-\infty}^{\infty} J_0 dx = \frac{1}{2}\int_{-\infty}^{\infty}(J+\bar J)dx = \frac{1}{2}(Q_0+\bar Q_0), \\ Q_{\rm ax} &= \int_{-\infty}^{\infty} J_1 dx = \frac{1}{2}\int_{-\infty}^{\infty}(J-\bar J)dx = \frac{1}{2}(Q_0-\bar Q_0) \end{aligned} \tag{3.4}$$

Taking into account the "chirality" of the free fields $\Phi(z)$, $\bar\Phi(\bar z)$, i.e., $\bar\partial\Phi(z) = \partial\bar\Phi(\bar z) = 0$, we have $\partial\Phi = 2\partial_x\Phi$ and $\bar\partial\bar\Phi = -2\partial_x\bar\Phi$. Therefore all charges $Q_{\rm vec}$, $Q_{\rm axial}$, Q_0, $\bar Q_0$ are defined in terms of the asymptotics of Φ and $\bar\Phi$ at $x\to\pm\infty$ (and fixed t, say $t=0$):

$$\begin{aligned} Q_0 &= 2\int_{-\infty}^{\infty}\partial_x\Phi dx = 2\left(\Phi(\infty)-\Phi(-\infty)\right), \\ \bar Q_0 &= -2\int_{-\infty}^{\infty}\partial_x\bar\Phi dx = -2\left(\bar\Phi(\infty)-\bar\Phi(-\infty)\right) \end{aligned} \tag{3.5}$$

We next observe that the second terms of J and $\bar J$ given by eq. (3.1) are in fact components of the global U(1)-current $I^u_\mu = (I, \bar I)$, i.e.,

$$\begin{aligned} I^u &= 2i\beta_0^2\psi_u\partial\chi_u e^{i\beta_0(R_u-\varphi_1)}, \\ \bar I^u &= -2i\beta_0^2\chi_u\bar\partial\psi_u e^{i\beta_0(R_u-\varphi_1)}, \qquad \bar\partial I^u + \partial\bar I^u = 0, \end{aligned} \tag{3.6}$$

generated by the following global U(1)-transformations (ϵ= const):

$$\psi'_u = \psi_u e^{i\beta_0^2\epsilon}, \qquad \chi'_u = \chi_u e^{-i\beta_0^2\epsilon}, \qquad R'_u = R_u, \qquad \varphi'_l = \varphi_l \tag{3.7}$$

As one might expect, its charge $Q^{\rm el}_u = \frac{1}{2}\int_{-\infty}^{\infty}(I+\bar I)dx$ is not independent of Q_0 and $\bar Q_0$:

$$Q^{\rm el}_u = \frac{2n}{n+1}Q^u_R - \frac{1}{2}(Q_0-\bar Q_0), \qquad Q^u_R = \beta_0\int_{-\infty}^{\infty}\partial_x R_u dx \tag{3.8}$$

due to the relation

$$\frac{1}{2}(J-\bar J) = \frac{2n\beta_0}{n+1}\partial_x R_u - \frac{1}{2}(I+\bar I) \tag{3.9}$$

between the currents J, $\bar J$, I, $\bar I$ and the topological current $J^\mu_{R_u} = \beta_0\epsilon^{\mu\nu}\partial_\nu R_u$, encoded in eq. (3.1). It reflects the fact that the transformation (3.7) is a particular case $w = \bar w = -\epsilon$ of eq. (1.6).

As we have shown in section 2.1, to each ungauged G_0-IM with local $G_0^0 \subset G_0$ symmetry, one can make in correspondence a new G_0/G_0^0-gauged IM, by gauge fixing G_0^0. The procedure of elimination of the extra G_0^0-field degrees of freedom (R_u for $G_0^0 = {\rm U}(1)$) consists in imposing constraints on the G_0^0-chiral currents ($J(z)$ and $\bar J(z)$):

$$\begin{aligned} J_g &= {\rm Tr}\left((g_0^f)^{-1}\partial g_0^f\lambda_1\cdot H\right) \\ &= \frac{2n}{n+1}\beta_0\partial R_g\left(1+\beta_0^2\frac{n+1}{2n}\psi_g\chi_g e^{-i\beta_0\varphi_1}\right) - 2i\beta_0^2\psi_g\partial\chi_g e^{-i\beta_0\varphi_1} = 0 \\ \bar J_g &= {\rm Tr}\left(\bar\partial g_0^f(g_0^f)^{-1}\lambda_1\cdot H\right) \\ &= \frac{2n}{n+1}\beta_0\bar\partial R_g\left(1+\beta_0^2\frac{n+1}{2n}\psi_g\chi_g e^{-i\beta_0\varphi_1}\right) - 2i\beta_0^2\chi_g\bar\partial\psi_g e^{-i\beta_0\varphi_1} = 0 \end{aligned} \tag{3.10}$$

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The relation between the field variables $g_0^u(\psi_u, \chi_u, R_u, \varphi_l) \in G_0$ of the ungauged IM (1.2)

$$g_0^u = e^{\chi_u E^{(0)}_{-\alpha_1}} e^{\beta(\lambda_1 \cdot H^{(0)} R_u + \sum_{i=1}^{n-1} \varphi_i h^{(0)}_{i+1})} e^{\psi_u E^{(0)}_{\alpha_1}}$$

and of the gauged IM (1.10) $g_0^f(\psi_g, \chi_g, \varphi_l) \in G_0/G_0^0$

$$g_0^f = e^{\chi_g E^{(0)}_{-\alpha_1}} e^{\beta \sum_{i=1}^{n-1} \varphi_i h^{(0)}_{i+1}} e^{\psi_g E^{(0)}_{\alpha_1}}$$

is given by

$$g_0^u = e^{\beta \bar{w}(\bar{z}) \lambda_1 \cdot H^{(0)}} \alpha_0 g_0^f \alpha_0 e^{\beta w(z) \lambda_1 \cdot H^{(0)}}, \qquad \alpha_0 = e^{\frac{1}{2}\beta \lambda_1 \cdot H^{(0)} R_g} \tag{3.11}$$

In components we find the following relations:

$$\begin{aligned}
R_u &= R_g + \beta_0(w + \bar{w}), \qquad \varphi_l^u = \varphi_l^g = \varphi_l, \\
\psi_u &= \psi_g e^{-\frac{1}{2} i \beta_0 R_g - i\beta_0^2 w}, \qquad \chi_u = \chi_g e^{-\frac{1}{2} i \beta_0 R_g - i\beta_0^2 \bar{w}}, \qquad \theta_g = \frac{1}{2i\beta_0} \ln\left(\frac{\chi_g}{\psi_g}\right) \\
\theta_u &= \theta_g - \frac{1}{2}\beta_0(w - \bar{w}), \qquad \theta_u = \frac{1}{2i\beta_0} \ln\left(\frac{\chi_u}{\psi_u}\right)
\end{aligned} \tag{3.12}$$

They reflect (a) the axial gauge fixing of chiral $\mathrm{U}(1)_L \otimes \mathrm{U}(1)_R$ symmetry and (b) the specific choice of the representative g_0^f of the coset G_0/G_0^0 (i.e. of the nonlocal field R_g) such that the $\mathcal{L}^g_{p=1}$ (1.10) is local in ψ_g, χ_g, i.e., independent of R_g. It is instructive to verify that by gauge transforming R_g, ψ_g and χ_g according to (3.12) with

$$w = \frac{1}{\beta_0^2}\left(\frac{n+1}{2n}\right)\Phi, \qquad \bar{w} = \frac{1}{\beta_0^2}\left(\frac{n+1}{2n}\right)\bar{\Phi}$$

one recover eq. (3.1) starting from the constraints (3.10) and vice versa.

The form of the global U(1) current $I^g_\mu = (I^g, \bar{I}^g)$ of the gauged IM (1.10)

$$I^g = 2i\beta_0^2 \frac{\psi_g \partial \chi_g e^{-i\beta_0\varphi_1}}{1 + \beta_0^2 \frac{n+1}{2n} \psi_g \chi_g e^{-i\beta_0\varphi_1}}, \qquad \bar{I}^g = -2i\beta_0^2 \frac{\chi_g \bar{\partial} \psi_g e^{-i\beta_0\varphi_1}}{1 + \beta_0^2 \frac{n+1}{2n} \psi_g \chi_g e^{-i\beta_0\varphi_1}} \tag{3.13}$$

i.e., $\bar{\partial} I^g + \partial \bar{I}^g = 0$, suggests that the constraints (3.10) (i.e. the gauge fixing conditions) can be considered as a requirement of the proportionality of the electric current I^g_μ (3.13) and the R_g-topological current $J^{R_g}_\mu = \beta_0 \epsilon_{\mu\nu} \partial^\nu R_g$, i.e.,

$$I^g_\mu = \frac{2n}{n+1} J^{R_g}_\mu \tag{3.14}$$

Note that in the ungauged IM (1.2) this relation is replaced by the more general one (3.9)

$$I^u_\mu = \epsilon_{\mu\nu}\left(\frac{2n}{n+1}\partial^\nu R_u - J^\nu\right) \tag{3.15}$$

and therefore I^u_μ and $J^{R_u}_\mu$ are independent due to the contribution of J^ν. The relation (3.12) establised between the fields of the gauged and ungauged IMs allows us to relate the corresponding U(1) and topological currents and charges of both models:

$$\begin{aligned}
Q^u_R &= Q^g_R + \frac{n+1}{4n}(Q_0 - \bar{Q}_0) = Q^g_R + \frac{n+1}{2n} Q_{\mathrm{ax}} \\
Q^u_\theta &= Q^g_\theta + \frac{n+1}{8n}(Q_0 + \bar{Q}_0) = Q^g_\theta + \frac{n+1}{4n} Q_{\mathrm{vec}}
\end{aligned} \tag{3.16}$$

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where we have introduced the topological charge Q_θ related to the field $\theta_g = \frac{1}{2i\beta_0}\ln(\chi_g/\psi_g)$. Replacing Q_R^u in eq. (3.8) we find that

$$Q_u^{\rm el} = \frac{2n}{n+1} Q_R^g = Q_g^{\rm el} \tag{3.17}$$

i.e., the global U(1) charges (of the currents I_u^μ and I_g^μ) do coincide. This leads to the conclusion that one can determine the charge spectrum of the ungauged IM in terms of the charges Q_R^g, Q_θ^g, $Q_g^{\rm el}$ (and $Q_{\rm top}^{\varphi_l} = \beta_0 \int_{-\infty}^{\infty} \partial_x \varphi_l dx$) of the gauged IM and the $J, \bar{J}$ charges ($Q_{\rm vec}$, $Q_{\rm ax}$ or Q_0, $\bar{Q}_0$), i.e., the asymptotics of Φ, $\bar{\Phi}$.

3.2 Nonconformal GKO coset construction

The relation between the fields (g_0^u, g_0^f and $\Phi, \bar{\Phi}$), currents and charges (3.15), (3.16) and (3.17) of the ungauged G_0-IM, the gauged G_0/G_0^0-IM and the chiral U(1)-CFT addresess the question whether the stress-tensors (and the energies) of these theories are related in similar manner. Having in mind that for the conformal limits[4] of the considered IMs (1.2) and (1.10) according to Goddard-Kent-Olive (GKO) coset construction [14], we have

$$T_{G_0}^{\rm CFT} = T_{G_0/G_0^0}^{\rm CFT} + T_{G_0^0}^{\rm CFT} \tag{3.18}$$

one might expect that an appropriate nonconformal extension of the GKO formula (3.18) to take place.

We start the derivation of the integrable models analog of eq. (3.18) by calculating the ungauged IM stress-tensor

$$\begin{aligned} T^u &= \frac{1}{2}\eta_{ij}\partial\varphi_i\partial\varphi_j + \frac{n}{2(n+1)}(\partial R_u)^2 + \partial\chi_u\partial\psi_u e^{i\beta_0(R_u-\varphi_1)} + V_u \\ \bar{T}^u &= T^u(\partial\to\bar{\partial}), \qquad T_{00}^u = \frac{1}{2}(T^u+\bar{T}^u), \qquad T_{01}^u = \frac{1}{2}(T^u-\bar{T}^u) \end{aligned} \tag{3.19}$$

We next substitute R_u, ψ_u and χ_u in T^u, taking into account eqs. (3.12) and the constraints (3.10) as well. By straightforward simplifications we realize that T^u can be written in terms of the gauged IM fields ψ_g, χ_g and φ_l together with the U(1)-CFT currents $J = \beta_0^2\frac{2n}{n+1}\partial w$ and $\bar{J} = \beta_0^2\frac{2n}{n+1}\bar{\partial}\bar{w}$ only, i.e.,

$$T^u = \frac{1}{2}\eta_{ij}\partial\varphi_i\partial\varphi_j + \frac{\partial\chi_g\partial\psi_g}{\Delta}e^{-\beta\varphi_1} + V_g + \frac{\beta_0^2 n}{2(n+1)}(\partial w)^2 \tag{3.20}$$

where $\Delta = 1+\beta^2\frac{n+1}{2n}\psi_g\chi_g e^{-\beta\varphi_1}$. Finally, we remind that the stress-tensor of the gauged IM, derived from its lagrangian (1.10) has the form

$$T^g = \frac{1}{2}\eta_{ij}\partial\varphi_i\partial\varphi_j + \frac{\partial\chi_g\partial\psi_g}{\Delta}e^{-\beta\varphi_1} + V_g$$

Therefore the *nonconformal* version of the GKO formula (3.18) is given by

$$T_{G_0}^u = T_{G_0/G_0^0}^g + T_{G_0^0}^{\rm CFT}, \qquad \bar{T}_{G_0}^u = \bar{T}_{G_0/G_0^0}^g + \bar{T}_{G_0^0}^{\rm CFT} \tag{3.21}$$

[4] I.e., negleting the perturbations $V_u^{\rm pert} = \frac{m^2}{\beta^2}\left(e^{\beta(\varphi_1+\varphi_{n-1})}(1+\beta^2\psi_u\chi_u e^{\beta(R_u-\varphi_1)}) - n\right)$ and $V_g^{\rm pert}$.

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or equivalently

$$T^u_{00} = T^g_{00} + T^{\rm CFT}_{00}\,, \qquad \bar{T}^u_{00} = \bar{T}^g_{00} + \bar{T}^{\rm CFT}_{00} \tag{3.22}$$

where $T^{\rm CFT}, \bar{T}^{\rm CFT}$ are the U(1)-CFT stress-tensors:

$$T^{\rm CFT} = \frac{\beta_0^2 n}{2(n+1)}(\partial w)^2\,, \qquad \bar{T}^{\rm CFT} = \frac{\beta_0^2 n}{2(n+1)}(\bar{\partial}\bar{w})^2 \tag{3.23}$$

Note that the $T^{\rm CFT}, \bar{T}^{\rm CFT}$ are chiral, i.e., $\bar{\partial}T^{\rm CFT} = \partial\bar{T}^{\rm CFT} = 0$ but $T^u, T^g, \bar{T}^u, \bar{T}^g$ are not, since the corresponding gauged and ungauged IMs are not conformal invariant.

The formal spliting of the G_0-ungauged IM fields, currents and stress-tensor in U(1)-CFT and gauged G_0/G_0^0-IM parts cannot be considered as an indication of a direct sum, since certain properties of the U(1)-CFT (b.c.'s of $w, \bar{w}$) depend on the gauged IM b.c.'s in the way indicated by the interaction terms of the ungauged IM, as we shall show in subsection 3.3.

3.3 Vacua, boundary conditions and discrete symmetries

The potential V_u of the ungauged IM (1.10) for imaginary coupling $\beta = i\beta_0$ manifest n-distinct zeroes:

$$\varphi_l^{(N)} = \frac{2\pi}{\beta_0}\frac{lN}{n}\,, \qquad \psi_u\chi_u = 0\,, \qquad R_u = \beta_0 a_R\,, \qquad \theta_u = \beta_0 a_\theta \tag{3.24}$$

where $N = 0, \pm 1, \ldots, \pm(n-1) \bmod n$, $l = 1, 2, \ldots, n-1$ and a_R, a_θ are real parameters. They represent the constant vacua solutions (i.e. $E_{\rm vac} = 0$, all charges $Q^{\rm vac}_{\rm all} = 0$) of the eq. of motion (2.28). The vacua values and the boundary conditions of the *massless* fields R_u and θ_u remain undefined by the $V_u = 0$ condition.[5] As usually the global symmetries of the model ($Z_2 \otimes Z_n$ and $\mathrm{U}(1)_{\rm vector} \otimes \mathrm{U}(1)_{\rm axial}$ in our case (1.2)) determine the complete vacua structure and allowed b.c.'s of all the fields. The Z_n-group that leave $\mathcal{L}^u_{p=1}$ (1.2) invariant acts as follows:

$$\begin{aligned} \varphi'_l &= \varphi_l + \frac{2\pi}{\beta_0}\frac{lN}{n}\,, \qquad R'_u = R_u + \frac{2\pi}{\beta_0}\left(s_R + \frac{N-q-\bar{q}}{n}\right) \\ \psi'_u &= \psi_u e^{\pi i\left(\frac{2q}{n}+s_1\right)}\,, \qquad \chi'_u = \chi_u e^{\pi i\left(\frac{2\bar{q}}{n}+s_2\right)}\,, \qquad \theta'_u = \theta_u + \frac{\pi}{\beta_0}\left(\frac{\bar{q}-q}{n} + s_\theta\right), \end{aligned} \tag{3.25}$$

where q, $\bar{q} = 0, \pm 1, \ldots \pm(n-1)$ and s_1, s_2, $2s_R = s_1 + s_2$, s_R, $s_\theta = s_2 - s_1$ are integers. The Z_n-charges of fields ψ_u, χ_u and $e^{\beta\varphi_l}$ are given by q, $\bar{q}$ and $N \bmod n$ respectively . One can further combine the above Z_n with the CP-transformation ($Px = -x$, $P\partial = \bar{\partial}$, $P^2 = 1$):

$$\varphi''_l = \varphi'_l\,, \qquad R''_u = R_u\,, \qquad \psi''_u = \chi_u\,, \qquad \chi''_u = \psi_u \tag{3.26}$$

into the larger diedral group D_n, requiring that ψ_u and χ_u are conjugated, i.e.,

$$\bar{q} = n - q \tag{3.27}$$

[5] Due to the fact that $\partial\tilde{V}_u/\partial R_u = \partial\tilde{V}_u/\partial\theta_u = 0$, (i.e. R_u and θ_u are flat directions) as one can see by suitable change of variables $\psi_u = \tilde{\psi}_u e^{-\frac{1}{2}\beta R_u}$, $\chi_u = \tilde{\chi}_u e^{-\frac{1}{2}\beta R_u}$ and $V_u \to \tilde{V}_u$.

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and $w(t-x) \leftrightarrow \bar{w}(t+x)$. Indeed by considering only chiral (say, left) U(1) transformations ($w \neq 0, \bar{w} = 0$) one breaks the CP-invariance ($\bar{q} = 0$, but $q \neq n$) and eq. (3.26) does not take place in this case. Completing the discussion about the relation between global symmetries and the vacua solutions we have also to mention the following global $\mathrm{U}(1)_{\mathrm{vector}} \otimes \mathrm{U}(1)_{\mathrm{axial}}$ symmetries of $\mathcal{L}^u_{p=1}$ (1.2):

- vector U(1)

$$\begin{aligned} \psi'_u &= e^{-i\beta_0^2 a_\theta}\psi_u\,, \qquad & \chi'_u &= e^{i\beta_0^2 a_\theta}\chi_u\,, \\ R'_u &= R_u\,, & \varphi'_l &= \varphi_l\,, \qquad \theta'_u = \theta_u + \beta_0 a_\theta \end{aligned} \tag{3.28}$$

- axial U(1)

$$\begin{aligned} \psi''_u &= e^{-i\frac{1}{2}\beta_0^2 a_R}\psi_u\,, \qquad & \chi''_u &= e^{-i\frac{1}{2}\beta_0^2 a_R}\chi_u\,, \\ R''_u &= R_u + \beta_0 a_R\,, & \varphi''_l &= \varphi_l\,, \qquad \theta''_u = \theta_u \end{aligned} \tag{3.29}$$

Therefore the allowed b.c.'s for the fields at $x \to \pm\infty$ are given by

$$\begin{aligned} \varphi_l^{(N)}(\pm\infty) &= \frac{2\pi}{\beta_0}\frac{lN_\pm}{n}\,, \qquad & R_u(\pm\infty) &= \beta_0 a_R^\pm + \frac{2\pi}{\beta_0}\left(s_R^\pm + \frac{N_\pm - q_\pm - \bar{q}_\pm}{n}\right), \\ \psi_u\chi_u(\pm\infty) &= 0, & \theta_u(\pm\infty) &= \beta_0 a_\theta^\pm + \frac{\pi}{\beta_0}\left(s_\theta^\pm + \frac{\bar{q}_\pm - q_\pm}{n}\right) \end{aligned} \tag{3.30}$$

Together with the vacua sector (defined by $D_n \otimes \mathrm{U}(1)_{\mathrm{vector}} \otimes \mathrm{U}(1)_{\mathrm{axial}}$), the ungauged IM (1.2) (in contrast to the gauged (1.10)) admits a new *conformal sector*. Due to chiral U(1) symmetry (1.6) its equations of motion (2.28) have conformal (1-D string-like) solutions

$$\varphi_l^{(N)} = \frac{2\pi}{\beta_0}\frac{lN}{n}\,, \qquad \psi_u\chi_u = 0\,, \qquad \theta_u^{\mathrm{CFT}} = \frac{\beta_0}{2}(\bar{w} - w)\,, \qquad R_u^{CFT} = \beta_0(w + \bar{w}) \tag{3.31}$$

with nonvanishing energy $E^{\mathrm{CFT}} = \pm P^{\mathrm{CFT}} \neq 0$ and charges $(Q_0, \bar{Q}_0 \neq 0)$ (3.5). The complete description of the U(1) CFT representing the conformal sector of the IM (1.2), requires the knowledge of the b.c.'s for the free fields $\Phi = \frac{2n\beta_0^2}{n+1}w$ and $\bar{\Phi} = \frac{2n\beta_0^2}{n+1}\bar{w}$. The question to be answered is whether one can uniquely determine the $\Phi, \bar{\Phi}$ b.c.'s from the relation (3.12) in terms of the b.c.'s of the fields of the ungauged IM ($\psi_u, \chi_u, R_u, \varphi_l$ given by eq. (3.30)) and of the gauged ones, $\psi_g, \chi_g, \varphi_l$ (see ref. [9]). We first consider the Z_n-transformations of w and $\bar{w}$ (and $\Phi, \bar{\Phi}$). According to eq. (3.12) we have,

$$e^{i\beta_0^2 w} = \frac{\psi_g}{\psi_u}e^{-i\frac{1}{2}\beta_0 R_g}\,, \qquad e^{i\beta_0^2\bar{w}} = \frac{\chi_g}{\chi_u}e^{-i\frac{1}{2}\beta_0 R_g}$$

and therefore the $w, \bar{w}$ properties are a consequence of the ψ_u, χ_u-transformation (3.25) and of the ψ_g, χ_g, R_g ones [9]:

$$\psi'_g = \psi_g e^{i\pi(\frac{N}{n} + \tilde{s}_1)}\,, \qquad \chi'_g = \chi_g e^{i\pi(\frac{N}{n} + \tilde{s}_2)}\,, \qquad R'_g = R_g\,, \qquad \tilde{s}_1 + \tilde{s}_2 = 2\tilde{s} \tag{3.32}$$

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where $\tilde{s}_1, \tilde{s}_2$ and $\tilde{s}$ are integers ($R_g = \frac{n}{n+1}R$ in the notation of ref. [9]). The result

$$w' = w + \frac{\pi}{\beta_0^2}\left(s + \frac{N-2q}{n}\right), \qquad \bar{w}' = \bar{w} + \frac{\pi}{\beta_0^2}\left(\bar{s} + \frac{N-2\bar{q}}{n}\right), \tag{3.33}$$
$$s = \tilde{s}_1 - s_1\,, \qquad \bar{s} = \tilde{s}_2 - s_2$$

is consistent with the R_u and θ_u transformations (3.25) under the identification

$$s = s_R + s_\theta\,, \qquad \bar{s} = s_R - s_\theta \tag{3.34}$$

Note that the particular case of left movers, i.e., $\bar{w} = 0$ and $w \neq 0$ takes place when the Z_n-charge $\bar{q}$ of χ_u is half of the φ_1-charge: $\bar{q} = N/2$ and $\tilde{s}_2 = s_2$.

The transformation properties (3.33) of $w, \bar{w}$ allow us to single out an important class of b.c.'s (at $x \to \pm\infty$) for the U(1)-CFT, namely,

$$\Phi(\pm\infty) = \frac{2\pi}{n+1}(ns_\pm + N_\pm - 2q_\pm)\,, \qquad \bar{\Phi}(\pm\infty) = \frac{2\pi}{n+1}(n\bar{s}_\pm + N_\pm - 2\bar{q}_\pm) \tag{3.35}$$

They give rise to *topological* CFT solitons of vortex type constructed in the next section 3.4. Observe that for these CFT solutions with b.c.'s (3.35) (i.e. interpolating between two coformal vacua $s_+, q_+, N_+ \to s_-, q_-, N_-$), the charges $Q_0, \bar{Q}_0$ (and $Q_{\rm ax}, Q_{\rm vec}$) (3.5) of the U(1) CFT takes the form:

$$\begin{aligned}
Q_0 &= \frac{4\pi n}{n+1}\left(s + \frac{j_w}{n}\right), \\
s &= s_+ - s_-\,, \qquad j_w = j_\varphi - 2j_q\,, \qquad j_\varphi = N_+ - N_-\,, \qquad j_q = q_+ - q_-\,, \\
\bar{Q}_0 &= -\frac{4\pi n}{n+1}\left(\bar{s} + \frac{j_{\bar{w}}}{n}\right), \qquad \bar{s} = \bar{s}_+ - \bar{s}_-\,, \qquad j_{\bar{w}} = j_\varphi - 2j_{\bar{q}}\,, \qquad j_{\bar{q}} = \bar{q}_+ - \bar{q}_- \\
s, \bar{s} &\in Z\,, \qquad j_\varphi, j_q, j_{\bar{q}} = 0, \pm 1, \ldots, \pm(n-1) \bmod n
\end{aligned} \tag{3.36}$$

The integers $s, \bar{s}$ denote the winding numbers of w and $\bar{w}$, i.e. the number of times w winds S^1 of radius $r_0 = \frac{1}{2\beta_0^2}$ when x is running from $-\infty$ to ∞. The integers j_w and $j_{\bar{w}}$ are the Z_n-charges of the vertices $V_w^{s,j_w} = e^{2i\beta_0^2 w}$ (or $V_{\bar{w}}^{\bar{s},j_{\bar{w}}} = e^{2i\beta_0^2\bar{w}}$). It is worthwhile to mention the relation of Q_0 and $\bar{Q}_0$ with the topological charges of the conformal fields $R^{\rm CFT} = \beta_0(w + \bar{w})$ and $\theta_R^{\rm CFT} = \frac{\beta_0}{2}(w - \bar{w})$, namely:

$$\begin{aligned}
Q_R^{\rm CFT} &= \beta_0 \int_{-\infty}^{\infty} \partial_x R^{\rm CFT} dx = \frac{n+1}{4n}(Q_0 - \bar{Q}_0)\,, \\
Q_\theta^{\rm CFT} &= \beta_0 \int_{-\infty}^{\infty} \partial_x \theta^{\rm CFT} dx = \frac{n+1}{8n}(Q_0 + \bar{Q}_0)
\end{aligned}$$

We next remind the vacua structure of the *gauged* IM (1.10) [9]:

$$\varphi_l^{(N)}(\pm\infty) = \frac{2\pi}{\beta_0}\frac{lN_\pm}{n}\,, \qquad \psi_g\chi_g(\pm\infty) = 0\,, \qquad \theta_g^{(L)}(\pm\infty) = \frac{\pi L_\pm}{2\beta_0}\,.$$

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The corresponding U(1) charged topological g-solitons (≡ gauged solitons) are characterized by their topological (j_φ, j_θ) and electric $j_{\rm el}$ charges,

$$\begin{aligned} Q^{\rm el} &= \frac{2n}{n+1}\beta_0 \int_{-\infty}^{\infty} \partial_x R_g dx = \beta_0^2 j_{\rm el}\,, \\ Q^g_\theta &= \beta_0 \int_{-\infty}^{\infty} \partial_x \theta_g dx = \frac{\pi j_\theta}{2}\,, \qquad\qquad j_\theta = L_+ - L_-\,, \\ Q^{top}_{\varphi_l} &= \frac{2n}{\beta_0} \int_{-\infty}^{\infty} \partial_x \varphi_l dx = \frac{4\pi l}{\beta_0^2} j_\varphi\,, \qquad j_\varphi = N_+ - N_- \bmod n \end{aligned} \tag{3.37}$$

where $j_\theta = 0$ for 1-solitons and $j_\theta \neq 0$ for charged breathers [9, 10]. The CFT dressing of such g-solitons according to eq. (3.12) (with w and $\bar{w}$ having specific b.c.'s (3.35)) maps each g-soliton to topological soliton (or string) of the ungauged IM (1.2) (called u-soliton), carrying the charges of both CFT and g-solitons:

$$(j_{\rm el}, j_\varphi, j_\theta | s, j_w; \bar{s}, j_{\bar{w}})$$

The one u-solitons are finite energy topological solutions of the IM (1.2) that interpolate between two different ungauged vacua

$$(j_\varphi | s, j_w; \bar{s}, j_{\bar{w}}) \tag{3.38}$$

($j_\theta = 0$ for 1-solitons). It becomes clear that the U(1)-CFT provides each g-vacua (j_φ, j_θ) with an infinite tower of conformal "states" (3.38) called u-vacua. The origin of its structure is in the allowed b.c.'s (3.30) for the IM (1.2).

3.4 U(1) CFT solitons

We are interested in a specific class of topological solutions of U(1) CFT with finite energy (real and positive) and having eqs. (3.35) as b.c.'s for $\Phi, \bar{\Phi}$ (and $w, \bar{w}$). The angular nature of $w_0 = 2\beta_0^2 w$ (and $\bar{w}_0 = 2\beta_0^2 \bar{w}$), i.e., of $R_0^{\rm CFT} = 2\beta_0 R^{\rm CFT}$, is an indication that $w_0, \bar{w}_0$ we seek represent the map of 2-D Minkowski space M_2 to the torus T_2:

$$M_2 \to S_1^{r_0} \otimes S_1^{r_0}\,, \qquad r_0 = \frac{1}{2\beta_0^2}$$

with certain discontinuities (branch cuts) allowed. Hence they should satisfy 2-D Poisson equation

$$\partial_\rho \partial_{\bar{\rho}} w_0 = \sum \alpha_i \delta^{(2)}(\rho - \rho_i)\,, \qquad \sum \alpha_i = 0 \tag{3.39}$$

where α_i are static charges localized at ρ_i and $\rho = e^{a_0 z}$, $\bar{\rho} = e^{a_0 \bar{z}}$ denote the new coordinates. The problem is quite similar to the vortex solutions of 2-D euclidean U(1) CFT [41] with "magnetic operators" $V_w^{s,0}(z_i) = e^{2i\beta_0^2 w(z_i)}$ creating discontinuity $2\pi s$ at $\rho = \rho_i$. It turns out that the simplest solution with all the required properties is given by the (twisted) Cayley transform:

$$w^{top} = -i\delta ln\left(\frac{e^{a_0 z} + i}{e^{a_0 z} - i}\right), \qquad \bar{w}^{top} = -i\bar{\delta}\ln\left(\frac{e^{a_0 \bar{z}} + i}{e^{a_0 \bar{z}} - i}\right) \tag{3.40}$$

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with

$$\delta = \frac{1}{\beta_0^2}\left(s + \frac{j_w}{n}\right), \qquad \bar{\delta} = \frac{1}{\beta_0^2}\left(\bar{s} + \frac{j_{\bar{w}}}{n}\right) \tag{3.41}$$

and a_0 is an arbitrary infrared (IR) scale. In the pure winding sector ($j_w = 0$, i.e., $j_q = \frac{j_\varphi}{2}$) we have

$$w_0^{\rm top}(\infty) = 0\,, \qquad w_0^{\rm top}(-\infty) = 2\pi s$$

which confirms the fact that $w_0^{\rm top}$ maps the infinite interval $(-\infty, \infty)$ to a circle. The energy of the left-moving solitons ($\bar{w}_0 = 0$) takes the form (see eq. (3.23)):

$$E^{\rm CFT}_{L-{\rm sol}} = \int_\infty^\infty T^{\rm CFT}(z)dx = \frac{2n\beta_0^2}{n+1}\int_\infty^\infty (\partial_x w^{\rm top})^2 dx = \frac{4n}{(n+1)\beta_0^2}\left(s + \frac{j_w}{n}\right)^2 |a_0| \tag{3.42}$$

The dimensionless quantity $\frac{E^{\rm CFT}_{L-{\rm sol}}}{|a_0|} = \Delta$ coincides with the conformal dimension of the vortex operator V_w^{s,j_w} with topological charge

$$Q_0 = \frac{4\pi n}{n+1}\left(s + \frac{j_w}{n}\right) \tag{3.43}$$

i.e. $\Delta = \frac{(n+1)}{(2\pi\beta_0)^2 n} Q_0^2$. The general solution of eq. (3.39) (with b.c.'s (3.35)) includes also an arbitrary holomorphic (i.e. string oscillators) part $w_{\rm str}$, i.e.,

$$w = w_{\rm top} + w_{\rm str}\,, \qquad w_{\rm str}(\pm\infty) = 0 \tag{3.44}$$

It has the same charge (3.43) as $w_{\rm top}$ but its energy acquires "string" contributions from $w_{\rm str}$:

$$E_{L-{\rm string}} = \frac{2n}{n+1}\left(\frac{2}{\beta_0^2}\left(s + \frac{j_w}{n}\right)^2 |a_0| + \beta_0^2 \sum_{l\neq 0} Q_l Q_{-l}\right) \tag{3.45}$$

where $\partial w_{\rm str} = \sum_{l\neq 0} Q_l z^l$.

We should mention that $w_{\rm top}$ given by eq. (3.40) is not the most general real non-homegeneous solution of eq. (3.39). One can construct one parameter family $w^\alpha_{\rm top}$ of such solutions with the same b.c.'s (3.35) but with arbitrary position of the branch cut:

$$w^\alpha_{\rm top} = -i\delta_\alpha \ln\left(\frac{e^{a_0 z} + e^{i\alpha}}{e^{a_0 z} + e^{-i\alpha}}\right), \qquad \delta_\alpha = \frac{(s + j_w/n)}{\beta_0^2}\left(\frac{\pi}{2\alpha}\right) \tag{3.46}$$

which for $\alpha = \pi/2$ concides with (3.40). By construction they carry the same topological charge (3.43), but their energy is α-dependent:

$$E^{\rm CFT}_{L-{\rm sol}}(\alpha) = \frac{4n}{(n+1)\beta_0^2}\left(s + \frac{j_w}{n}\right)^2 |a_0| \left(\frac{\pi}{2\alpha}\right)^2 \left(1 - \alpha\frac{\cos(\alpha)}{\sin(\alpha)}\right) \tag{3.47}$$

and positive for, say, $\pi/4 < \alpha < \pi$.

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The left (and right) solitons are massless by construction, since $E_L = P_L$ (or $E_R = -P_R$ for the right ones). One could have however a nontrivial scattering of left and right solitons of rapidities $b_L - b_R \sim 0$,

$$E_L = P_L = \frac{1}{2} M_0 e^{b_L}\,, \qquad E_R = -P_R = \frac{1}{2} M_0 e^{-b_R}$$

leading to massive poles at intermediate (crossover) energy scale $\sim M_0^2$ that spoils the infrared scale (and conformal) invariance [42]. Kinematically such possibility indeed exists,

$$M_0^2 = 4 E_L^{\rm CFT} E_R^{\rm CFT}\,, \qquad b_L - b_R \sim 0$$

and it has been studied in the context of the marginal perturbation of SU(2) WZW model in ref. [42]. We assume that such phenomenon (existence of preferable scale, breaking conformal symmetry at intermediate energies) takes place in the IM under consideration, i.e. together with massless solitons we also have the massive left-right solitons (for $\alpha = \frac{\pi}{2}$)

$$\begin{aligned} M_0 &= \frac{8n}{(n+1)\beta_0^2} \left(s + \frac{j_w}{n}\right) \left(\bar{s} + \frac{j_{\bar{w}}}{n}\right) |a_0| \\ Q_0 &= \frac{4\pi n}{n+1} \left(s + \frac{j_w}{n}\right), \qquad \bar{Q}_0 = -\frac{4\pi n}{n+1} \left(\bar{s} + \frac{j_{\bar{w}}}{n}\right) \end{aligned} \tag{3.48}$$

We remind that the CP-invariance (3.27) imposes

$$s = -\bar{s}\,, \qquad j_w + j_{\bar{w}} = 2(j_\varphi - n)\,, \qquad j_q + j_{\bar{q}} = n$$

The proof of the conjecture of appearence of massive solitons at certain intermediate scale M_0 requires the construction of the corresponding left-right S-matrices which is out of the scope of this paper.

3.5 Spectrum of the ungauged 1-solitons

The explicit construction of the u-solitons is based on eq. (3.12), replacing w and $\bar{w}$ by the U(1)-CFT solitons (3.40) and $\psi_g, \chi_g, R_g = \frac{n+1}{n} R$ — by the corresponding 1-soliton solutions of the gauged IM (1.10) (see ref. [9, eqs. (4.5)–(4.10)]). The nonconformal GKO formula (3.21) allows us to calculate the energies (and masses) of the ungauged IM 1-solitons in terms of the energies of the constituent gauged and CFT solitons. For the left-u-solitons we find

$$\begin{aligned} &E_u = E_g + E_{L-\rm sol}^{\rm CFT}\,, \qquad P_u = P_g + P_{L-\rm sol}^{\rm CFT}\,, \qquad E_L^{\rm CFT} = P_L^{\rm CFT}\,, \\ &E_g = M_g \cosh b\,, \qquad P_g = -M_g \sinh b\,, \qquad M_u^2 = M_g \left(M_g + 2E_{L-\rm sol}^{\rm CFT} e^b\right) \end{aligned} \tag{3.49}$$

where b is the g-soliton velocity and M_g its mass [9]

$$M_g = \frac{4\mu}{\beta_0^2} n \left| \sin\left(\frac{\beta_0^2 j_{\rm el} - 4\pi j_\varphi}{4n}\right)\right| \tag{3.50}$$

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Due to the arbitrariness of the $|a_0|$ scale, we can introduce a new scale parameter m_0, such that $|a_0| = m_0 e^{-b}$. Then the mass M_u of the ungauged 1-soliton takes the form

$$\left(\frac{M_u}{m_0}\right)^2 = \left(\frac{M_g}{m_0}\right)\left(\frac{M_g}{m_0} + \frac{8n}{(n+1)\beta_0^2}\left(s + \frac{j_w}{n}\right)^2\right) \tag{3.51}$$

which in fact determines the dimensionless mass-ratios M_u/m_0 only. According to eqs. (3.16), (3.17) and (3.43) the charge spectrum of the left u-soliton is given by:

$$\begin{aligned} Q_\theta^u &= \frac{n+1}{8n} Q_0 = \frac{\pi}{2}\left(s + \frac{j_w}{n}\right), \qquad Q_\theta^g = 0\,, \\ Q_R^u &= \frac{n+1}{n}\left(\beta_0^2 j_{\rm el} + \frac{2\pi n}{n+1}\left(s + \frac{j_w}{n}\right)\right), \\ Q_{\rm el}^u &= Q_{\rm el}^g = \beta_0^2 j_{\rm el}\,, \qquad j_{\rm el} = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{3.52}$$

Similar formulae take place for the right-u-soliton ($w = 0, \bar{w} \neq 0$).

The spectrum of the left-right u-soliton combines the energies, masses and charges of the g-soliton [9] with the left-right soliton (3.48):

$$\begin{aligned} E_u^{L-R} &= E_g + E_{L-\rm sol}^{\rm CFT} + E_{R-\rm sol}^{\rm CFT}\,, \qquad & M_u^{L-R} &= M_g + M_0 \\ Q_\theta^u &= \frac{n+1}{8n}(Q_0 + \bar{Q}_0)\,, & Q_R^u &= \frac{n+1}{2n}\left(Q_{\rm el}^u + \frac{1}{2}(Q_0 - \bar{Q}_0)\right) \end{aligned} \tag{3.53}$$

Together with charged topological u-solitons representing stable strong coupling particles one can also have u-strings represented by eqs. (3.12), but with $w_{\rm top}, \bar{w}_{\rm top}$ of eq. (3.40) replaced by $w = w_{\rm top} + w_{\rm str}$ (3.44). They have the same charge spectrum as the u-solitons (but with infinite set of charges Q_k and $\bar{Q}_k$ added) and the energy spectrum including the string oscillator part:

$$E_u^{\rm string} = E_g + E_{L-\rm string} + E_{R-\rm string} \tag{3.54}$$

where $E_{L-\rm string}, (E_{R-\rm string})$ are given by (3.45).

3.6 θ-terms, dyonic effects and spectral flows

Similarly to the gauged IM [9], one can add to the ungauged IM lagrangian (1.2) certain topological θ-terms, i.e., $\mathcal{L}_u^{\rm impr} = \mathcal{L}_u + \delta\mathcal{L}_u^{\rm top}$ with $\delta\mathcal{L}_u^{\rm top}$ given by

$$\begin{aligned} \delta\mathcal{L}_u^{\rm top} = \frac{\beta}{8\pi^2}\Bigg(&\sum_{k=1}^{n-1} \nu_k^\varphi \epsilon^{\mu\nu}\partial_\mu\varphi_k\partial_\nu \ln\left(\frac{\chi_u}{\psi_u}\right) + 2\pi\beta\sum_{k=1}^{n-1}\tilde{\nu}_k\epsilon^{\mu\nu}\partial_\mu\varphi_k\partial_\nu R_u + \\ &+ \pi\nu^R\epsilon^{\mu\nu}\partial_\mu R_u\partial_\nu \ln\left(\frac{\chi_u}{\psi_u}\right)\Bigg), \end{aligned} \tag{3.55}$$

($\nu_k^\varphi, \tilde{\nu}_k, \nu^R$ are real parameters). They do not change the equations of motion, but contributes to the charges $Q_{\rm el}, Q_0, \bar{Q}_0 \to Q_{\rm el}^{\rm impr}, Q_0^{\rm impr}, \bar{Q}_0^{\rm impr}$. For example, the electric current I_μ^u (3.6) (generated by the global U(1) transformations (3.7)) acquires extra terms,

$$I_\mu^{u,{\rm impr}} = I_\mu^u - \frac{\beta_0^3}{4\pi^2}\epsilon_{\mu\nu}\left(\sum_{k=1}^{n-1}\nu_k^\varphi\partial^\nu\varphi_k + 2\pi\nu^R\partial^\nu R_u\right) \tag{3.56}$$

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Then, say for left-u-solitons we have

$$Q_{\rm el}^{\rm impr} = Q_{\rm el} - \frac{\nu\beta_0^2}{2\pi} j_\varphi - \frac{\nu^R}{2\pi} Q_R^u\,, \qquad Q_0^{\rm impr} = Q_0 - \frac{\tilde\nu\beta_0^2}{\pi} j_\varphi - \frac{\nu^R\beta_0^2}{\pi} Q_\theta^u\,,$$
$$Q_R^u = \frac{n+1}{2n}\left(Q_{\rm el} + \frac{1}{2}Q_0\right), \qquad Q_\theta^u = \frac{n+1}{8n} Q_0 \tag{3.57}$$

where we have introduced ν and $\tilde\nu$ as follows

$$\nu = \frac{1}{n}\sum_{k=1}^{n-1} k\nu_k^\varphi\,, \qquad \tilde\nu = \frac{1}{n}\sum_{k=1}^{n-1} k\tilde\nu_k^\varphi$$

Taking into account eq. (3.43) and that $Q_{\rm el}^{\rm impr} = \beta_0^2 j_{\rm el}$ is quantized semiclassically [9], we derive the improved charge spectrum

$$Q_0^{\rm impr} = \frac{4\pi n}{n+1}\left(1 - \frac{\beta_0^2\nu^R(n+1)}{8\pi n}\right)\left(s + \frac{j_w}{n}\right) - \frac{\beta_0^2\tilde\nu}{\pi} j_\varphi - \frac{\beta_0^2\nu^R}{2} j_\theta\,,$$
$$Q_{\rm el} = \frac{\beta_0^2\left(j_{\rm el} + \frac{\nu}{2\pi} j_\varphi + \frac{\nu^R}{2}(s + \frac{j_w}{n})\right)}{1 - \frac{\beta_0^2\nu^R}{4\pi n}(n+1)} \tag{3.58}$$

The interpretation of the parameters $\nu_k^\varphi, \tilde\nu_k$ and ν^R as external constant magnetic fields is similar to one already presented in refs. [9, 23].

Note that the shift in Q_0 by $\tilde\nu j_\varphi$ and $\nu_R j_\theta$ that gives $Q_0^{\rm impr}$ (3.57) (and similar for $\bar Q_0^{\rm impr}$) appears to be the *nonconformal* analog of the spectral flow [43]

$$\tilde J_3^0 = J_3^0 + \frac{k}{4}\tilde\omega \tag{3.59}$$

($\tilde\omega$ is an integer) playing an important role in the description of AdS_3-CFT (i.e. the representations of $\mathrm{SL}(2,\mathbb{R})$ current algebra).

3.7 Solitons of deformed $\mathrm{SL}(2,\mathbb{R})$-WZW model

As we have mentioned in section 2.3, the particular case $n = 1$ and β real of the IM (1.2) represents integrable deformation of the $\mathrm{SL}(2,\mathbb{R})$-WZW model described by the lagrangian (2.34). The problem of the vacua structure and soliton solutions of this IM should be considered separately by the following three reasons:

- its potential $V_u^{n=1} = m^2\psi_u\chi_u e^{\beta R_u}$ has not distinct zeroes and therefore one cannot expect to have topological solitons.
- for real β the $R^{\rm CFT}$ (and $w, \bar w$) b.c.'s are not periodic and as a consequence the U(1)-CFT solitons are not topological too.
- contrary to the $n \geq 2$ IMs the g-solitons of the $n = 1$ gauged IM (i.e. Lund-Regge [44, 45]) which are an important ingredient of the u-solitons are also nontopological.

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Therefore we have to make certain modifications in the arguments of section 3.3–3.5 in order to derive the spectrum of such nontopological u-solitons. All the relations between currents, charges and stress-tensors of the gauged and ungauged IM of section 3.1 and 3.2 are still valid for $n = 1$ and β real. For example, the field relation (3.12) now reads

$$\begin{aligned} R_u &= R_g + \beta(w + \bar{w})\,, \qquad \theta_u = \theta_g - \frac{\beta}{2}(w - \bar{w})\,, \\ \psi_u &= \psi_g e^{-\frac{\beta}{2}R_g - \beta^2 w}\,, \qquad \chi_u = \chi_g e^{-\frac{\beta}{2}R_g - \beta^2 \bar{w}} \end{aligned} \tag{3.60}$$

Due to the fact that this case ($n = 1$, β real)[6] we have no analog of the Z_n discrete symmetry (3.25), the b.c.'s of the R_u (and $w, \bar{w}$) are determined by the global $\mathrm{U}(1)_{\mathrm{vector}} \otimes \mathrm{U}(1)_{\mathrm{axial}}$ transformations (3.28) and (3.29) only, i.e.,

$$\begin{aligned} \psi'_u &= \psi_u e^{-\frac{\beta^2}{2}a_R}\,, & \chi'_u &= \chi_u e^{-\frac{\beta^2}{2}a_R}\,, & R'_u &= R_u + \beta a_R\,, & \theta'_u &= \theta_u, \\ \psi''_u &= \psi_u e^{-\beta^2 a_\theta}\,, & \chi''_u &= \chi_u e^{\beta^2 a_\theta}\,, & R''_u &= R_u\,, & \theta'_u &= \theta_u + \beta a_\theta \end{aligned} \tag{3.61}$$

Therefore the allowed b.c.'s for w and $\bar{w}$ have the form

$$w' = w + a_w\,, \qquad \bar{w}' = \bar{w} + \bar{a}_w\,, \qquad a_w + \bar{a}_w = a_R\,, \qquad \bar{a}_w - a_w = 2a_\theta \tag{3.62}$$

where $a_w, \bar{a}_w, a_R, a_\theta$ are arbitrary real constants. As a consequence of (3.62) we can chose

$$w(\pm\infty) = a_w^{\pm}\,, \qquad \bar{w}(\pm\infty) = \bar{a}_w^{\pm} \tag{3.63}$$

which determine the spectrum of the U(1)-charges $Q_0, \bar{Q}_0$, namely

$$Q_0 = 2\beta^2(a_w^+ - a_w^-) \equiv \frac{2\pi a}{\beta^2}\,, \qquad \bar{Q}_0 = -2\beta^2(\bar{a}_w^+ - \bar{a}_w^-) \equiv -\frac{2\pi\bar{a}}{\beta^2} \tag{3.64}$$

Since $2\beta R^{\mathrm{CFT}}$ and $2\beta^2 w$ are *not* angular variables in the $\mathrm{SL}(2,\mathbb{R})$ case (and the charge spectrum is continuous) the w_{top} and $\bar{w}_{\mathrm{top}}$ in the form (3.40) have no topological meaning. They map M_2 into a rectangle (with finite area $\pi^2\delta\bar{\delta}$) instead the torus as it is in the case of imaginary $\beta = i\beta_0$. Our choice of w_{top} and $\bar{w}_{\mathrm{top}}$ is again in the form (3.40), but with $\delta, \bar{\delta}$-continuous, i.e.,

$$\delta = \frac{2a}{\beta^2}\,, \qquad \bar{\delta} = -\frac{2\bar{a}}{\beta^2} \tag{3.65}$$

It is dictated by eqs. (3.39) and indeed ensures the required discontinuities. The energy of such solutions ("nontopological" CFT-solitons) is finite and positive:

$$E^{\mathrm{CFT}}_{L-\mathrm{sol}} = \frac{8a^2}{\beta^2}|a_0|\,, \qquad E^{\mathrm{CFT}}_{R-\mathrm{sol}} = \frac{8\bar{a}^2}{\beta^2}|a_0| \tag{3.66}$$

[6] For imaginary $\beta = i\beta_0$ and $n = 1$, i.e., SU(2)-WZW, however the following Z-transformation

$$\psi'_u = \psi_u e^{i\pi s}\,, \qquad \chi'_u = \chi_u e^{-i\pi s}\,, \qquad R'_u = R_u - \frac{2\pi}{\beta_0}s_R$$

takes place ($\psi^*_u = \chi_u$ in SU(2) case).

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According to eq. (3.60) (and similarly to the generic $n \geq 2$ case of section 3.5) the 1-solitons of this deformed $SL(2,\mathbb{R})$-WZW are a composition of the above "CFT-solitons" and the 1-solitons (nontopological) of the gauged $n = 1$ IM (i.e. Lund-Regge model [44]). Their semi classical spectrum [45] is given by

$$M_g = \frac{4m}{\beta^2} \sin\left(\frac{\beta^2 j_{\rm el}}{4}\right), \qquad Q^g_{\rm el} = \beta^2 j_{\rm el}\,, \qquad j_{\rm el} = 0, \pm 1, \pm 2, \ldots \tag{3.67}$$

Hence the spectrum of the nontopological left 1-solitons of the IM (2.34) has the form

$$Q^u_R = \beta^2 j_{\rm el} + \frac{a}{\beta^2}\,, \qquad Q^u_\theta = \frac{a}{2\beta^2}\,, \qquad M_u^2 = M_g\left(M_g + \frac{16a^2}{\beta^2} m_0\right) \tag{3.68}$$

The energy of the corresponding $n = 1$ string solutions is quite similar to the generic n formula (3.45), (3.54) with $E_{L-\rm sol}$ replaced by (3.66) and E_g — with the Lund-Regge soliton energy.

It is worthwile to mention that in the SU(2) case ($\beta = i\beta_0$, R_u with periodic b.c.'s and $\psi^*_u = \chi_u$) the charges $Q_0, \bar{Q}_0$ are quantized, i.e. $a = s \in Z$ and the corresponding CFT-solitons are topological and stable. Similarly to the $n \geq 2$ IMs (1.2) the deformed SU(2) WZW admits four kinds of 1-soliton solutions:

- massless topological U(1) CFT solitons and strings
- massive solitons of the gauged (Lund-Regge)IM
- massive composite left (and right) u-solitons
- massive left-right u-solitons

Whether the composed left-u-solitons of deformed $SL(2,\mathbb{R})$ WZW with spectrum (3.68) represent stable strong coupling particles is an open question. It is clear however that the corresponding u-solitons of the SU(2) model are indeed topologicaly stable.

4. Multicharged IMs with local and global symmetries

Among the vast family of dyonic IMs introduced in section 2, we have choosen the simplest $A^{(1)}_n (p = 1)$ IM (1.2) (with one local U(1) symmetry) in order to demonstrate how the spectrum of its u-solitons can be realized in terms of the charges, energies , etc of the g-solitons of the gauged IM (1.10) and certain U(1) CFT solitons (3.40). It is natural to address the question of whether the established reduction of the ungauged IM properties to the corresponding gauged IM combined with U(1) CFT (with specific b.c.'s) takes place for generic multicharged IMs, i.e., $\mathcal{G}^0_0 = \mathrm{U}(1)^l$. Our main attention in the present section is concentrated on the $\mathcal{G}^0_0 = \mathrm{U}(1) \otimes \mathrm{U}(1)$ multicharged IM (2.14) and particularly on the intermediate IM (2.18) with one local and one global U(1) symmetries.

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4.1 Conserved charges and GKO energy spliting

The chiral $\mathrm{U}(1)\otimes\mathrm{U}(1)$ conserved currents

$$J_a = \mathrm{Tr}\left(g_0^{-1}\partial g_0 \lambda_a\cdot H^{(0)}\right)\,, \qquad \bar J_a = \mathrm{Tr}\left(\bar\partial g_0 g_0^{-1}\lambda_a\cdot H^{(0)}\right), \qquad a=1,n$$

($g_0\in G_0$ is given by eq. (2.13)) of the ungauged IM (2.14), generated by the transformations (2.16), (2.17), take the following explicit form:

$$\begin{aligned}
J_1 &= \frac{2\beta_0}{n+1}(n\partial R_1^u+\partial R_n^u) - i\beta_0^2\psi_1^u\partial\chi_1^u e^{i\beta_0(R_1^u-\varphi_1)}\,,\\
J_n &= \frac{2\beta_0}{n+1}(\partial R_1^u+n\partial R_n^u) - i\beta_0^2\psi_n^u\partial\chi_n^u e^{i\beta_0(R_n^u-\varphi_{n-2})}\,,\\
\bar J_1 &= \frac{2\beta_0}{n+1}(n\bar\partial R_1^u+\bar\partial R_n^u) - i\beta_0^2\chi_1^u\bar\partial\psi_1^u e^{i\beta_0(R_1^u-\varphi_1)}\,,\\
\bar J_n &= \frac{2\beta_0}{n+1}(\partial R_1^u+n\partial R_n^u) - i\beta_0^2\psi_n^u\partial\chi_n^u e^{i\beta_0(R_n^u-\varphi_{n-2})}
\end{aligned} \tag{4.1}$$

We denote their charges by

$$Q^a_{m+1} = \oint J_a(z)z^{-m-1}dz\,, \qquad \bar Q^a_{m+1} = \oint \bar J_a(\bar z)\bar z^{-m-1}d\bar z\,, \qquad m\in Z \tag{4.2}$$

Similarly to the U(1) case (3.8), the topological charges

$$Q^a_{R_u} = \beta_0\int_{-\infty}^{\infty}\partial_x R_a^u dx\,, \qquad a=1,n, \tag{4.3}$$

the global $\mathrm{U}(1)\otimes\mathrm{U}(1)$ electric charges $Q_a^{el,u}$:

$$\begin{aligned}
Q_1^{el,u} &= i\beta_0^2\int_{-\infty}^{\infty}(\psi_1^u\partial\chi_1^u-\chi_1^u\partial\psi_1^u)e^{i\beta_0(R_1^u-\varphi_1)}dx\\
Q_n^{el,u} &= i\beta_0^2\int_{-\infty}^{\infty}(\psi_n^u\partial\chi_n^u-\chi_n^u\partial\psi_n^u)e^{i\beta_0(R_n^u-\varphi_{n-2})}dx
\end{aligned} \tag{4.4}$$

and zero modes $Q_0^a,\bar Q_0^a$ of the chiral currents $J^a,\bar J^a$ are related as follows

$$\begin{aligned}
Q_1^{el,u} &= \frac{2}{n+1}\left(nQ^{(1)}_{R_u}+Q^{(n)}_{R_u}\right)-\frac{1}{2}\left(Q_0^{(1)}-\bar Q_0^{(1)}\right)\\
Q_n^{el,u} &= \frac{2}{n+1}\left(Q^{(1)}_{R_u}+nQ^{(n)}_{R_u}\right)-\frac{1}{2}\left(Q_0^{(n)}-\bar Q_0^{(n)}\right)
\end{aligned} \tag{4.5}$$

The fact that by construction the gauged IM (2.22) is a result of the gauge fixing of the local $\mathrm{U}(1)\otimes\mathrm{U}(1)$ symmetries (2.17) (in the way that $J_a=\bar J_a=0$) and of the consequent elimination of two degrees of freedom $R_a^u, a=1,n$ lead to the following relation between the gauged and ungauged fields appearing in (2.22) and in (2.14) respectively,

$$\begin{aligned}
R_a^u &= R_a^g+\beta_0(w_a+\bar w_a)\,, & \varphi_l^u &= \varphi_l^g=\varphi_l\,,\\
\psi_a^u &= \psi_a^g e^{-\frac{1}{2}i\beta_0 R_a^g - i\beta_0^2 w_a}\,, & \chi_a^u &= \chi_a^g e^{-\frac{1}{2}i\beta_0 R_a^g - i\beta_0^2\bar w_a}\,, \qquad a=1,n\\
\theta_a^u &= \theta_g^a-\frac{\beta_0}{2}(w_a-\bar w_a)\,, & \theta_a^u &= \frac{1}{2i\beta_0}\ln\left(\frac{\psi_a^u}{\chi_a^u}\right)
\end{aligned} \tag{4.6}$$

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The matrix form of these relations is given by eq. (2.24). As in the case of the U(1)-models of section 3, eqs. (4.6) are crucial in the construction of the solutions of the ungauged IM (2.14) in terms of the known solutions [23] of the gauged IM (2.22) and certain $\mathrm{U}(1)\otimes\mathrm{U}(1)$ -CFT solutions (i.e. specific $w_a, \bar{w}_a$). An important byproduct of eqs. (4.6) are the following relations between the charges (topological and Noether) of both models (2.14) and (2.22)

$$\begin{aligned}
Q^{(1)}_{R^u} &= Q^{(1)}_{Rg} + \frac{n}{4(n-1)}\left(Q^{(1)}_0 - \bar{Q}^{(1)}_0 - \frac{1}{n}\left(Q^{(n)}_0 - \bar{Q}^{(n)}_0\right)\right)\\
Q^{(n)}_{R^u} &= Q^{(n)}_{Rg} + \frac{n}{4(n-1)}\left(Q^{(n)}_0 - \bar{Q}^{(n)}_0 - \frac{1}{n}\left(Q^{(1)}_0 - \bar{Q}^{(1)}_0\right)\right)\\
Q^{(1)}_{\theta^u} &= Q^{(1)}_{\theta g} + \frac{n}{8(n-1)}\left(Q^{(1)}_0 + \bar{Q}^{(1)}_0 + \frac{1}{n}\left(Q^{(n)}_0 + \bar{Q}^{(n)}_0\right)\right)\\
Q^{(n)}_{\theta^u} &= Q^{(n)}_{\theta g} + \frac{n}{8(n-1)}\left(Q^{(n)}_0 + \bar{Q}^{(n)}_0 + \frac{1}{n}\left(Q^{(1)}_0 + \bar{Q}^{(1)}_0\right)\right)
\end{aligned} \tag{4.7}$$

where the charges $Q^a_0, \bar{Q}^a_0$ have been realized in terms of the asymptotics of the free fields $\Phi^a(z), \bar{\Phi}^a(\bar{z})$ (i.e., $J^a = \partial\Phi^a$, $\bar{J}^a = \bar{\partial}\bar{\Phi}^a$):

$$\begin{aligned}
Q^a_0 &= 2\int_{-\infty}^{\infty}\partial_x\Phi^a dx = 2(\Phi^a(\infty) - \Phi^a(-\infty))\,,\\
\bar{Q}^a_0 &= -2\int_{-\infty}^{\infty}\partial_x\bar{\Phi}^a dx = -2(\bar{\Phi}^a(\infty) - \bar{\Phi}^a(-\infty))
\end{aligned} \tag{4.8}$$

which are related to w_a and $\bar{w}_a$ as follows

$$\begin{aligned}
\Phi^{(1)} &= \frac{2\beta_0^2}{n+1}(nw_1 + w_n)\,, \qquad \bar{\Phi}^{(1)} = \frac{2\beta_0^2}{n+1}(n\bar{w}_1 + \bar{w}_n)\,,\\
\Phi^{(n)} &= \frac{2\beta_0^2}{n+1}(w_1 + nw_n)\,, \qquad \bar{\Phi}^{(n)} = \frac{2\beta_0^2}{n+1}(\bar{w}_1 + n\bar{w}_n)
\end{aligned} \tag{4.9}$$

Again, as in the U(1) case (3.17), the electric charges $Q^{el,u}_a$ and $Q^{el,g}_a$ of the ungauged and gauged IMs do coincide:

$$\begin{aligned}
Q^{el,u}_1 &= \frac{2}{n+1}\left(nQ^{(1)}_{Rg} + Q^{(n)}_{Rg}\right) = Q^{el,g}_1\,,\\
Q^{el,u}_n &= \frac{2}{n+1}\left(Q^{(1)}_{Rg} + nQ^{(n)}_{Rg}\right) = Q^{el,g}_n
\end{aligned} \tag{4.10}$$

as one can verify by substituting (4.7) in (4.5).

The generalization of the nonconformal GKO formulae (3.21) and (3.22) to the case of multicharged IMs (2.14) and (2.22) is straightforward. Substituting eqs. (4.6) in the ungauged IM stress-tensor:

$$\begin{aligned}
T^u_{p=2} = \frac{1}{2}\eta_{ij}\partial\varphi_i\partial\varphi_j + \partial\psi^u_1\partial\chi^u_1 e^{\beta(R^u_1-\varphi_1)} + \partial\psi^u_n\partial\chi^u_n e^{\beta(R^u_n-\varphi_{n-2})} +\\
+ \frac{n}{2(n+1)}\left((\partial R^u_1)^2 + (\partial R^u_n)^2 + \frac{2}{n}\partial R^u_1\partial R^u_n\right) + V_u
\end{aligned} \tag{4.11}$$

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and $\bar{T}^u_{p=2} = T^u_{p=2}(\partial \to \bar{\partial})$ and taking into account the constraints [23]

$$J(z) = \mathrm{Tr}\left(\left(g_0^f\right)^{-1} \partial g_0^f \lambda_a \cdot H^{(0)}\right) = \bar{J}(z) = \mathrm{Tr}\left(\bar{\partial} g_0^f \left(g_0^f\right)^{-1} \lambda_a \cdot H^{(0)}\right) = 0\,,$$

we realize that the following GKO-spliting

$$T^u_{p=2} = T^g_{p=2} + \beta_0^2 \frac{n}{2(n+1)} \left((\partial w_1)^2 + (\partial w_n)^2 + \frac{2}{n}(\partial w_1)(\partial w_n)\right) \tag{4.12}$$

takes place. We have denoted by $T^g_{p=2}$ the canonical stress-tensor of the gauged IM derived from its lagrangian (2.22),

$$\begin{aligned} T^g_{p=2} = {} & \frac{1}{2}\eta_{ij}\partial\varphi_i\partial\varphi_j + \\ & + \frac{1}{\Delta}\Bigg(\partial\psi_1^g\partial\chi_1^g e^{-\beta\varphi_1}\left(1 + \beta_0^2\frac{n}{2(n-1)}\psi_n^g\chi_n^g e^{-\beta\varphi_{n-2}}\right) + \\ & \quad + \partial\psi_n^g\partial\chi_n^g e^{-\beta\varphi_{n-2}}\left(1 + \beta_0^2\frac{n}{2(n-1)}\psi_1^g\chi_1^g e^{-\beta\varphi_1}\right) + \\ & \quad + \beta^2\frac{1}{2(n-1)}\left(\chi_1^g\psi_n^g\partial\psi_1^g\partial\chi_n^g + \chi_n^g\psi_1^g\partial\psi_n^g\partial\chi_1^g\right)e^{-\beta(\varphi_1+\varphi_{n+2})}\Bigg) + V_g \end{aligned} \tag{4.13}$$

One can further diagonalize the $\mathrm{U}(1)\otimes\mathrm{U}(1)$-CFT stress-tensor

$$T^{\mathrm{CFT}}_{p=2} = \beta_0^2\frac{n}{2(n+1)}\left((\partial w_1)^2 + (\partial w_n)^2 + \frac{2}{n}\partial w_1\partial w_n\right), \qquad \bar{\partial}T^{\mathrm{CFT}}_{p=2} = 0$$

by introducing new fields $w_\pm$:

$$w_\pm = \frac{1}{2}(w_1 \pm w_n)\,, \qquad \bar{w}_\pm = \frac{1}{2}(\bar{w}_1 \pm \bar{w}_n) \tag{4.14}$$

The result is:

$$T^{\mathrm{CFT}}_{p=2} = \beta_0^2\left((\partial w_+)^2 + \frac{n-1}{n+1}(\partial w_-)^2\right) = T^{\mathrm{CFT}}_+ + T^{\mathrm{CFT}}_-\,, \tag{4.15}$$

and the same for $\bar{T}^{\mathrm{CFT}}_{p=2} = T^{\mathrm{CFT}}_{p=2}(\partial \to \bar{\partial})$.

The intermediate IM (2.18), introduced in section 2, is a result of the gauge fixing of one of the local U(1) symmetries, say,

$$\begin{aligned} J_+ &= \frac{1}{2}\,\mathrm{Tr}\left(\left(g_0^{\mathrm{int}}\right)^{-1}\partial g_0^{\mathrm{int}}(\lambda_1+\lambda_n)\cdot H^{(0)}\right) = 0\,, \\ \bar{J}_+ &= \frac{1}{2}\,\mathrm{Tr}\left(\bar{\partial} g_0^{\mathrm{int}}\left(g_0^{\mathrm{int}}\right)^{-1}(\lambda_1+\lambda_n)\cdot H^{(0)}\right) = 0 \end{aligned} \tag{4.16}$$

of the ungauged IM (2.14). Its lagrangian is invariant under chiral U(1) (spanned by

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$\frac{1}{2}(\lambda_1-\lambda_n)\cdot H^{(0)})$ transformation (2.21) with conserved current $\bar\partial J_- = \partial \bar J_- = 0$

$$
\begin{aligned}
J_- &= \frac{1}{2}\,\mathrm{Tr}\left(\left(g_0^{\mathrm{int}}\right)^{-1}\partial g_0^{\mathrm{int}}\,(\lambda_1-\lambda_n)\cdot H^{(0)}\right)\\
&= 2\beta_0\left(\frac{n-1}{n+1}\right)\Delta_0\partial\bar R_u - i\beta_0^2\Bigg(\left(1+\frac{\beta^2}{2}\bar\psi_n\bar\chi_n e^{-i\beta_0(\bar R+\varphi_{n-2})}\right)\bar\psi_1\partial\bar\chi_1 e^{i\beta_0(\bar R-\varphi_1)} -\\
&\qquad - \left(1+\frac{\beta^2}{2}\bar\psi_1\bar\chi_1 e^{-i\beta_0(\varphi_1-\bar R)}\right)\bar\psi_n\partial\bar\chi_n e^{-i\beta_0(\bar R+\varphi_{n-2})}\Bigg)\\
\bar J_- &= J_-(\partial\to\bar\partial,\psi_a\to\chi_a) = \frac{1}{2}\,\mathrm{Tr}\left(\bar\partial g_0^{\mathrm{int}}\,\left(g_0^{\mathrm{int}}\right)^{-1}(\lambda_1-\lambda_n)\cdot H^{(0)}\right)
\end{aligned}
\tag{4.17}
$$

where

$$
J_- = \partial\Phi_-\,,\qquad \bar J = \partial\Phi_-\,,\qquad \Phi_- = 2\beta_0^2\frac{n-1}{n+1}w_-\,,\qquad \bar\Phi_- = 2\beta_0^2\frac{n-1}{n+1}\bar w_-\,. \tag{4.18}
$$

They are also invariant under global U(1) transformation

$$
\bar\psi_a' = \bar\psi_a e^{i\beta_0^2\epsilon_+}\,,\qquad \bar\chi_a' = \bar\chi_a e^{-i\beta_0^2\epsilon_+}\,,\qquad \bar R_u' = \bar R_u\,,\qquad \varphi_l' = \varphi_l \tag{4.19}
$$

We denote the charges of the J_- and $\bar J_-$ currents by

$$
Q^-_{m+1} = \oint J_-(z)z^{-m-1}dz\,,\qquad \bar Q^-_{m+1} = \oint \bar J_-(\bar z)\bar z^{-m-1}d\bar z\,,
$$

i.e., we have for their zero modes:

$$
Q_0^- = 2\int_{-\infty}^{\infty}\partial_x\Phi_- dx\,,\qquad \bar Q_0^- = -2\int_{-\infty}^{\infty}\partial_x\bar\Phi_- dx \tag{4.20}
$$

The global U(1)-charges are now given by

$$
Q_\pm^{\mathrm{el},u} = \frac{1}{2}\left(Q_1^{\mathrm{el},u}\pm Q_n^{\mathrm{el},u}\right) \tag{4.21}
$$

Taking into account eq. (4.10), we derive the following relation

$$
Q_+^{\mathrm{el},u} = 2Q_{\tilde R_g} = Q_+^{\mathrm{el},g}\,,\qquad Q_-^{\mathrm{el},u} = 2\frac{n-1}{n+1}Q_{\bar R_g} = Q_-^{\mathrm{el},g} \tag{4.22}
$$

between $Q_\pm^{\mathrm{el},u}$ and the topological charges

$$
\begin{aligned}
Q_{\tilde R_g} &= \beta_0\int_{-\infty}^{\infty}\partial_x\tilde R_g dx\,,\qquad \tilde R_g = \frac{1}{2}(R_1^g+R_2^g),\\
Q_{\bar R_g} &= \beta_0\int_{-\infty}^{\infty}\partial_x\bar R_g dx\,,\qquad \bar R_g = \frac{1}{2}(R_1^g-R_2^g)
\end{aligned}
\tag{4.23}
$$

Note that the topological charge $Q_{\bar R_u}$, i.e.,

$$
Q_{\bar R_u} = \beta_0\int_{-\infty}^{\infty}\partial_x\bar R_u dx
$$

is not proportional to the electric charge $Q_-^{\mathrm{el},u}$, due to the more general relation

$$
\frac{1}{2}(Q_0^- - \bar Q_0^-) = 2\left(\frac{n-1}{n+1}\right)Q_{\bar R_u} - Q_-^{\mathrm{el},u} \tag{4.24}
$$

which follows from eq. (4.17).

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According to eq. (2.20) the fields of the intermediate model $\bar{\psi}_a, \bar{\chi}_a, \bar{R}$ can be realized in terms of the fields of the ungauged IM $\psi_a^u, \chi_a^u, R_a^u$ and the free fields w_+ and $\bar{w}_+$:

$$\begin{aligned} \tilde{R}_u &= \tilde{R}_g^{\rm int} + \beta_0(w_+ + \bar{w}_+)\,, \qquad & \varphi_l^u &= \varphi_l^{\rm int}\,, \qquad & \bar{R}_u &= \bar{R}_u^{\rm int} = \bar{R}\,, \\ \psi_a^u &= \bar{\psi}_a e^{-i\beta_0^2 w_+ - \frac{i}{2}\beta_0 \tilde{R}_g}\,, & \chi_a^u &= \bar{\chi}_a e^{-i\beta_0^2 \bar{w}_+ - \frac{i}{2}\beta_0 \tilde{R}_g} \end{aligned} \tag{4.25}$$

Similarly we can relate the intermediate IM fields with those of the gauged IM (1.10) $\psi_a^g, \chi_a^g, R_a^g$, taking into account eqs. (4.25), (4.6) and (4.14):

$$\begin{aligned} \bar{R} &= \bar{R}_g + \beta_0(w_- + \bar{w}_-)\,, \qquad & \varphi_l^{\rm int} &= \varphi_l^g\,, \qquad & \bar{\psi}_1 &= \psi_1^g e^{-i\beta_0^2 w_- - \frac{i}{2}\beta_0 \bar{R}_g}\,, \\ \bar{\psi}_n &= \psi_n^g e^{i\beta_0^2 w_- + \frac{i}{2}\beta_0 \bar{R}_g}\,, & \bar{\chi}_1 &= \chi_1^g e^{-i\beta_0^2 w_- - \frac{i}{2}\beta_0 \bar{R}_g}\,, & & \\ \bar{\chi}_n &= \chi_n^g e^{i\beta_0^2 w_- + \frac{i}{2}\beta_0 \bar{R}_g}\,, & \bar{\theta}_a &= \theta_a^g \mp \frac{\beta_0}{2}(w_- - \bar{w}_-)\,, \qquad a = 1, n \end{aligned} \tag{4.26}$$

Therefore the topological charges of the intermediate IM can be realized in terms of the corresponding gauged IM charges and the J_- and $\bar{J}_-$ zero modes:

$$\begin{aligned} Q_{\bar{R}_u} &= Q_{\bar{R}_g} + \frac{n+1}{4(n-1)}(Q_0^- - \bar{Q}_0^-)\,, \\ Q_{\theta_u}^{(1)} &= Q_{\theta_g}^{(1)} - \frac{n+1}{8(n-1)}(Q_0^- + \bar{Q}_0^-)\,, \\ Q_{\theta_u}^{(n)} &= Q_{\theta_g}^{(n)} + \frac{n+1}{8(n-1)}(Q_0^- + \bar{Q}_0^-) \end{aligned} \tag{4.27}$$

Finally, the GKO energy spliting for the intermediate IM takes the form

$$T_{p=2}^u = T_{p=2}^{\rm int} + \beta_0^2(\partial w_+)^2\,, \qquad T_{p=2}^{\rm int} = T_{p=2}^g + \beta_0^2\left(\frac{n-1}{n+1}\right)(\partial w_-)^2 \tag{4.28}$$

The stress-tensor of the intermediate IM (2.18) is given by

$$\begin{aligned} T_{p=2}^{\rm int} = {} & \frac{1}{2}\eta_{ij}\partial\varphi_i\partial\varphi_j + \left(\frac{n-1}{n+1}\right)(\partial\bar{R})^2 + \\ & + \frac{1}{\Delta_0}\Bigg(\left(1 + \frac{\beta_0^2}{4}\bar{\psi}_n\bar{\chi}_n e^{-i\beta_o(\bar{R}+\varphi_{n-2})}\right) e^{i\beta_0(\bar{R}-\varphi_1)}\partial\bar{\psi}_1\partial\bar{\chi}_1 + \\ & \quad + \left(1 + \frac{\beta_0^2}{4}\bar{\psi}_1\bar{\chi}_1 e^{-i\beta_o(\bar{R}-\varphi_1)}\right) e^{i\beta_0(\bar{R}+\varphi_E n-2)}\partial\bar{\psi}_n\partial\bar{\chi}_n - \\ & \quad - \frac{\beta_0^2}{4}e^{-i\beta_0(\varphi_1+\varphi_{n-2})}\left(\bar{\psi}_1\bar{\chi}_n\partial\bar{\chi}_1\partial\bar{\psi}_n + \bar{\psi}_n\bar{\chi}_1\partial\bar{\chi}_n\partial\bar{\psi}_1\right)\Bigg) + V_{\rm int} \end{aligned} \tag{4.29}$$

and $T_{p=2}^u$ and $T_{p=2}^g$ by eqs. (4.11) and (4.13) respectively.

4.2 Vacua Structure and 1-solitons of the $\mathrm{U}(1) \otimes \mathrm{U}(1)$ ungauged IM

Together with the local (and global) $\mathrm{U}(1) \otimes \mathrm{U}(1)$ symmetries (described in section 4.1) the ungauged IM lagrangian (2.14) is also invariant under discrete $Z_2 \otimes Z_2 \otimes Z_{n-1}$ transformations (for imaginary coupling $\beta = i\beta_0$). The action of the Z_{n-1} group on the fields

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$\varphi_l, \psi^u_a, \chi^u_a$ and R^u_a is quite similar to the Z_n transformations (3.25) for the U(1) model (i.e. $p = 1$) (1.2). It is easy to check that $\mathcal{L}^u_{p=2}$ (2.14) remains invariant under the following Z_{n-1} transformation:

$$\begin{aligned}
\varphi'_l &= \varphi_l + \frac{2\pi}{\beta_0}\frac{lN}{n-1}\,, \qquad l = 1, 2, \ldots, n-2\,,\\
(R^u_1)' &= R^u_1 + \frac{2\pi}{\beta_0}\left(s^{(1)}_R + \frac{N - q_1 - \bar{q}_1}{r-1}\right),\\
(R^u_n)' &= R^u_n + \frac{2\pi}{\beta_0}\left(s^{(n)}_R - \frac{N - q_n - \bar{q}_n}{r-1}\right),\\
(\psi^u_1)' &= e^{\pi i(\frac{2q_1}{n-1} + s_1)}\psi^u_1\,, \qquad (\chi^u_1)' = e^{\pi i(\frac{2\bar{q}_1}{n-1} + \tilde{s}_1)}\chi^u_1\,,\\
(\psi^u_n)' &= e^{-\pi i(\frac{2q_n}{n-1} + s_n)}\psi^u_n\,, \qquad (\chi^u_n)' = e^{-\pi i(\frac{2\bar{q}_n}{n-1} + \tilde{s}_n)}\chi^u_n
\end{aligned} \tag{4.30}$$

where $s^{(a)}_R, s_a, \tilde{s}_a$, $(s_a \pm \tilde{s}_a = 2s^{(a)}_\pm)$ are integers and $q_a, \bar{q}_a, N = 0, 1, \ldots, n-2 \bmod(n-1)$ are the Z_{n-1} charges of the fields ψ^u_a, χ^u_a and $e^{i\beta_0\varphi_1}$ respectively. The first Z_2 acts as field reflections:

$$\begin{aligned}
&\varphi''_l = \varphi_{n-l-1}\,, \qquad (R^u_1)'' = R^u_n\,, \qquad (R^u_n)'' = R^u_1\,,\\
&(\psi^u_1)'' = \psi^u_n\,, \qquad (\psi^u_n)'' = \psi^u_1\,, \qquad (\chi^u_1)'' = \chi^u_n\,, \qquad (\chi^u_n)'' = \chi^u_1\,,
\end{aligned} \tag{4.31}$$

i.e. interchanging the Z_{n-1} charges of φ_1 and φ_{n-2}, ψ^u_1 and ψ^u_n, R^u_1 and R^u_n, etc.

$$N \to n - 1 - N\,, \qquad q_a \to n - 1 - q_a\,, \qquad \bar{q}_a \to n - 1 - \bar{q}_a$$

This extends the Z_{n-1} to the diedral group D_{n-1}. The other Z_2 group represents the following CP-transformations:

$$\begin{aligned}
&\varphi'''_l = \varphi_l\,, \qquad (R^u_a)''' = R^u_a, \qquad (\psi^u_a)''' = \chi^u_a\,, \qquad (\chi^u_a)''' = \psi^u_a\,,\\
&Px = -x\,, \qquad P\partial = \bar{\partial}\,, \qquad P^2 = 1
\end{aligned} \tag{4.32}$$

The vacuum solutions for the ungauged IM (2.14) are given by

$$\varphi^{(N)}_l = \left(\frac{2\pi}{\beta_0}\right)\frac{lN}{n-1}\,, \qquad \psi_a\chi_a = 0\,, \qquad R^u_a = \beta_0 a^a_R\,, \qquad \theta^u_a = \beta_0 a^a_\theta \tag{4.33}$$

For such field configuration the potential $V^u_{p=2}$ (2.15) shows $(n-1)$-distinct zeroes. By chiral $\mathrm{U}(1) \otimes \mathrm{U}(1)$ transformations (2.17) one map these constant vacua solutions into a special class of conformal invariant solutions

$$\begin{aligned}
\varphi^{(N)}_l &= \frac{2\pi}{\beta_0}\frac{lN}{n-1}\,, \qquad \psi^u_a\chi^u_a = 0\\
R^{u,CFT}_a &= \beta_0(w_a + \bar{w}_a)\,, \qquad \theta^{u,CFT}_a = \frac{\beta_0}{2}(w_a - \bar{w}_a)\,, \qquad a = 1, n
\end{aligned} \tag{4.34}$$

representing string (2-d free fields $w_a, \bar{w}_a$) in flat 2-d target space, i.e., $\mathrm{U}(1) \otimes \mathrm{U}(1)$-CFT. Among all possible string-like and particle-like finite energy ($E^{\mathrm{CFT}} = \pm P^{\mathrm{CFT}}$) solutions of this CFT, we seek for a special family of charged topological massless solitons. As in

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the case of U(1)-CFT (see section 3.3 and 3.4) crucial for the existence of such solutions are the nontrivial b.c. for $w_a, \bar{w}_a$, supported by certain discrete symmetries (Z_{n-1} in our case). It is not necessary to impose an appropriate b.c., since they are already encoded in eqs. (4.6) and (4.30). Taking into account the Z_{n-1} transformations of the gauged IM (2.14) fields [23]:

$$\begin{aligned}
(\psi_1^g)' &= e^{\pi i(\frac{N}{n-1}+s_1^g)}\psi_1^g\,, \qquad & (\chi_1^g)' &= e^{\pi i(\frac{N}{n-1}+\tilde{s}_1^g)}\chi_1^g\,,\\
(\psi_n^g)' &= e^{-\pi i(\frac{N}{n-1}+s_n^g)}\psi_n^g\,, \qquad & (\chi_n^g)' &= e^{-\pi i(\frac{N}{n-1}+\tilde{s}_n^g)}\chi_n^g\\
(R_a^g)' &= R_a^g\,, & \varphi_l' &= \varphi_l
\end{aligned} \tag{4.35}$$

we derive the following discrete transformations for $w_a, \bar{w}_a$,

$$w_a' = w_a + \frac{\pi}{\beta_0^2}\left(s_a + \epsilon_a\frac{N-2q_a}{n-1}\right), \qquad \bar{w}_a' = \bar{w}_a + \frac{\pi}{\beta_0^2}\left(\bar{s}_a + \epsilon_a\frac{N-2\bar{q}_a}{n-1}\right), \tag{4.36}$$

where $\epsilon_1 = -\epsilon_n = 1$. The Z_{n-1} properties (4.36) of $w_a, \bar{w}_a$ (and of the diagonal fields (4.14) $w^{\pm}$ and $\bar{w}^{\pm}$) allow us to distiguish an important class of b.c. for the U(1) $\otimes$ U(1)-CFTs free fields $\Phi^a(z)$ and $\bar{\Phi}^a(\bar{z})$ of eqs. (4.9):

$$\begin{aligned}
\Phi_+(\pm\infty) &= \pi\left(s_1^{\pm} + s_n^{\pm} + \frac{2(q_n^{\pm} - q_1^{\pm})}{n-1}\right),\\
\Phi_-(\pm\infty) &= \pi\frac{n-1}{n+1}\left(s_1^{\pm} - s_n^{\pm} + \frac{2(N_{\pm} - q_1^{\pm} - q_n^{\pm})}{n-1}\right)
\end{aligned} \tag{4.37}$$

where $\Phi_{\pm}$ are defined as follows

$$\begin{aligned}
\Phi_+ &= \frac{1}{2}\left(\Phi^{(1)} + \Phi^{(n)}\right) = 2\beta_0^2 w_+\,,\\
\Phi_- &= \frac{1}{2}\left(\Phi^{(1)} - \Phi^{(n)}\right) = 2\frac{n-1}{n+1}\beta_0^2 w_-\,,\\
J_{\pm} &= \partial\Phi_{\pm}\,, \qquad J_{\pm} = \frac{1}{2}(J_1 \pm J_n)
\end{aligned} \tag{4.38}$$

Therefore the CFT solutions interpolating between two vacua

$$(\Phi_+(\infty), \Phi_-(\infty)) \to (\Phi_+(-\infty), \Phi_-(-\infty))$$

carry nontrivial topological charges $Q_0^{\pm}$ and $\bar{Q}_0^{\pm}$

$$\begin{aligned}
Q_0^+ &= 2\int_{-\infty}^{\infty}\partial_x\Phi_+ dx = 4\pi\left(s_+ + \frac{j_w^-}{n-1}\right),\\
Q_0^- &= 2\int_{-\infty}^{\infty}\partial_x\Phi_- dx = 4\frac{n-1}{n+1}\pi\left(s_- + \frac{j_w^+}{n-1}\right),\\
\bar{Q}_0^+ &= -4\pi\left(\bar{s}_+ + \frac{j_{\bar{w}}^-}{n-1}\right), \qquad \bar{Q}_0^- = -4\pi\frac{n-1}{n+1}\left(\bar{s}_- + \frac{j_{\bar{w}}^+}{n-1}\right)
\end{aligned} \tag{4.39}$$

where $s_1 \pm s_n = 2s_{\pm}$, $2j_w^{\pm} = j_w^{(1)} \pm j_w^{(n)}$, $j_w^a = j_\varphi - 2j_q^a$, $j_\varphi = N_+ - N_-$, $j_q^a = q_+^a - q_-^a$. It becomes clear that the conformal vacua of the U(1) $\otimes$ U(1)-CFT are characterized by

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the winding numbers $s_\pm$ and $\bar{s}_\pm$ of the fields $2\beta_0^2 w_\pm$ and $2\beta_0^2 \bar{w}_\pm$ and by the Z_{n-1} charges $j_w^\pm$, $j_{\bar{w}}^\pm$ of the vertices $V_\pm^{s_\pm, j_w^\pm} = e^{2i\beta_0^2 w_\pm}$ and $\bar{V}_\pm^{\bar{s}_\pm, j_{\bar{w}}^\pm} = e^{2i\beta_0^2 \bar{w}_\pm}$. Similarly to the U(1)-case of section 3.4, the topological solitons of the U(1)$\otimes$U(1)-CFT with all the above properties are nothing but the map of 2-d dimension space M_2 to 4-torus $T_4 = (S_1^{r_0})^4, r_0 = \frac{1}{2\beta_0^2}$. Their explicit form is a generalization of the 1-soliton $w_{\rm top}$ (3.40) of U(1)-CFT:

$$\begin{aligned} w_\pm^{\rm top} &= -i\delta_\pm ln\left(\frac{e^{a_0 z}+i}{e^{a_0 z}-i}\right), \qquad \bar{w}_\pm^{\rm top} = -i\bar{\delta}_\pm ln\left(\frac{e^{a_0 \bar{z}}+i}{e^{a_0 \bar{z}}-i}\right), \\ \delta_\pm &= \frac{1}{\beta_0^2}\left(s_\pm + \frac{j_w^\mp}{n-1}\right), \qquad \bar{\delta}_\pm = \frac{1}{\beta_0^2}\left(\bar{s}_\pm + \frac{j_{\bar{w}}^\mp}{n-1}\right) \end{aligned} \tag{4.40}$$

We next calculate the energy $E_{L-{\rm sol}}^{p=2}$ of the left moving massless solitons (see eq. (4.15)):

$$\begin{aligned} E_{L-{\rm sol}}^{p=2} &= \int_{-\infty}^{\infty} T_{P=2}^{\rm CFT} dx = 4\beta_0^2 \int_{-\infty}^{\infty} \left((\partial_x w_+)^2 + \frac{n-1}{n+1}(\partial_x w_-)^2\right) dx \\ &= 8\beta_0^2 |a_0| \left(\delta_+^2 + \frac{n-1}{n+1}\delta_-^2\right) \end{aligned}$$

or explicitly:

$$\frac{E_{L-{\rm sol}}^{p=2}}{|a_0|} = \frac{8}{\beta_0^2}\left(\left(s_+ + \frac{j_w^-}{n-1}\right)^2 + \frac{n-1}{n+1}\left(s_- + \frac{j_w^+}{n-1}\right)^2\right) \tag{4.41}$$

One can also have finite energy string-like solutions with the same topological charges (4.39), but admiting arbitrary holomorphic parts $w_{\rm str}^\pm$ of $w^\pm$, i.e.,

$$w^\pm = w_\pm^{\rm top} + w_{\rm str}^\pm, \qquad w_{\rm str}^\pm(\pm\infty) = 0$$

The energy of such string solutions gets contributions from $\partial w_{\rm str}^\pm = \sum_{k\neq 0} Q_k^\pm z^k$:

$$E_{L-{\rm string}}^{p=2} = E_{L-{\rm sol}}^{p=2} + \beta_0^2 \sum_{k\neq 0}\left(Q_k^+ Q_{-k}^+ + \frac{n-1}{n+1} Q_k^- Q_{-k}^-\right) \tag{4.42}$$

We should mention that the most general particle-like 1-soliton is represented by the following two parameter family of solutions:

$$w_{\rm top}^\pm(\alpha_\pm) = -i\delta_\pm(\alpha_\pm)\ln\left(\frac{e^{a_0 z}+e^{i\alpha_\pm}}{e^{a_0 z}+e^{-i\alpha_\pm}}\right), \qquad \delta_\pm(\alpha_\pm) = \frac{\pi}{2\alpha_\pm}\delta_\pm \tag{4.43}$$

They have the same topological charges (4.39) as $w_{\rm top}^\pm(\alpha_\pm = \pi/2) = w_{\rm top}^\pm$, but different energies (for $\alpha_\pm \neq \pi/2$, $\pi/4 < \alpha_\pm < \pi$)

$$\begin{aligned} E_{L-{\rm sol}}^{p=2}(\alpha_+,\alpha_-) = \frac{2|a_0|\pi^2}{\beta_0^2}\Bigg(&\frac{1}{\alpha_+^2}\left(s_+ + \frac{j_w^-}{n-1}\right)^2 (1-\alpha_+\,{\rm cotg}(\alpha_+)) + \\ &+ \frac{1}{\alpha_-^2}\left(\frac{n-1}{n+1}\right)\left(s_- + \frac{j_w^+}{n-1}\right)^2 (1-\alpha_-\,{\rm cotg}(\alpha_-))\Bigg) \end{aligned} \tag{4.44}$$

which coincides with (4.41) for $\alpha_\pm = \pi/2$.

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Given the 1-solitons [23] of the gauged IM (2.22) and the U(1)⊗U(1)-CFT topological massless solitons (4.40) and (4.43) we can construct the ungauged 1-solitons of the IM (2.14) according to eqs. (4.6). Then the GKO formula (4.12) allows us to calculate the energy of say, left-u-solitons (i.e., $w^{\pm} \neq 0, \bar{w}^{\pm} = 0$):

$$\begin{aligned} E_u^{p=2} &= E_g^{p=2} + E_{L-\mathrm{sol}}^{\mathrm{CFT}}(p=2)\,, \\ P_u^{p=2} &= P_g^{p=2} + P_{L-\mathrm{sol}}^{\mathrm{CFT}}(p=2)\,, \qquad E_L^{\mathrm{CFT}} = P_L^{\mathrm{CFT}}\,, \\ E_g^{p=2} &= M_g^{p=2}\cosh(b)\,, \qquad\qquad P_g^{p=2} = -M_g^{p=2}\sinh(b) \end{aligned} \tag{4.45}$$

where b is the velocity of the gauged IM 1-solitons and M_g their mass [23]:

$$M_g^{p=2} = \frac{4m(n-1)}{\beta_0^2}\left|\sin\frac{4\pi j_\varphi - \beta_0^2 j_{\mathrm{el}}}{4(n-1)}\right|$$

For $|a_0| = m_0 e^{-b}$, the mass of u-solitons is given by

$$\left(\frac{M_u^{p=2}}{m_0}\right)^2 = \frac{M_g^{p=2}}{m_0}\left(\frac{M_g^{p=2}}{m_0} + 2\frac{E_{L-\mathrm{sol}}^{\mathrm{CFT}}(p=2)}{m_0}\right) \tag{4.46}$$

Taking into account the charge relations (4.5), (4.7) and (4.10) we have established in section 4.1, we derive the charge spectrum of the left-u-soliton:

$$\begin{aligned} Q_0^+ &= 4\pi\left(s_+ + \frac{j_w^-}{n-1}\right), \qquad Q_0^- = 4\pi\frac{n-1}{n+1}\left(s_- + \frac{j_w^+}{n-1}\right), \\ Q_{\mathrm{el}}^- &= \frac{1}{2}\left(Q_1^{\mathrm{el}} - Q_n^{\mathrm{el}}\right) = \frac{\beta_0^2}{2}j_{\mathrm{el}}\,, \qquad Q_{\mathrm{el}}^+ = \frac{1}{2}\left(Q_1^{\mathrm{el}} + Q_n^{\mathrm{el}}\right) = \frac{\beta_0^2}{2}\delta_0\,, \\ Q_{R_u}^- &= \frac{1}{2}\left(Q_{R_u}^{(1)} - Q_{R_u}^{(n)}\right) = \frac{n+1}{4(n-1)}\left(\beta_0^2 j_{\mathrm{el}} + Q_0^-\right), \\ Q_{R_u}^+ &= \frac{1}{2}\left(Q_{R_u}^{(1)} + Q_{R_u}^{(n)}\right) = \frac{1}{4}\left(2\beta_0^2\delta_0 + Q_0^+\right), \\ Q_{\theta_u}^+ &= \frac{n+1}{4(n-1)}Q_0^+\,, \qquad Q_{\theta_u}^- = \frac{1}{4}Q_0^- \end{aligned} \tag{4.47}$$

where $Q_{\mathrm{el}}^{\pm}$ (i.e. $j_{\mathrm{el}} \in Z$ and $\delta_0 \in R$) represent the semiclassical electric charges of the gauged 1-solitons [23].

4.3 Soliton spectrum of the intermediate IM

The intermediate model (2.18) arises from the ungauged IM (2.14) by axial gauge fixing one of the chiral U(1) symmetries, namely by imposing the constraints $J_+ = \bar{J}_+ = 0$ (see eq. (4.16)) and keeping J_- and $\bar{J}_-$ unconstrained. The same way we have constructed u-solitons as conformal dressing by w_{top}^a (4.40) of the 1-solitons [23] of the gauged IM (2.22), the intermediate 1-solitons can be constructed by U(1) CFT-dressing of the same g-solitons by w_{top}^- and $\bar{w}_{\mathrm{top}}^-$ according to eqs. (4.26). The relations between the solitons of the gauged, intermediate and ungauged IMs, established in section 4.1 provide an alternative construction of these int-solitons as a reduction of the u-solitons by requiring that $w^+ = \bar{w}^+ = 0$, i.e.,

$$q_1 = q_n = q\,, \qquad \bar{q}_1 = \bar{q}_n = \bar{q}\,, \qquad s_+ = \bar{s}_- = 0$$

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In order to make transparent the properties of these solitons (and the origin of the b.c.'s for $w^-, \bar{w}^-$) we shall derive the vacua structure of the IM (2.18) and its discrete symmetries independently of the relations with the ungauged IM (2.14) . The Z_{n-1} transformations that leave invariant the lagrangian (2.18) have the form:

$$\begin{aligned}
&\bar{\psi}'_a = \bar{\psi}_a e^{i\pi\epsilon_a(\frac{2q}{n-1}+s_a)}\,, \qquad \bar{\chi}'_a = \bar{\chi}_a e^{i\pi\epsilon_a(\frac{2\bar{q}}{n-1}+\bar{s}_a)}\,, \qquad \epsilon_1 = -\epsilon_n = 1\,, \qquad s_a \pm \bar{s}_a = 2L_a\,,\\
&\bar{R}' = \bar{R} + \frac{2\pi}{\beta_0}\left(s_R + \frac{N-q-\bar{q}}{n-1}\right), \qquad\qquad \varphi' = \varphi_l + \frac{2\pi}{\beta_0}\frac{lN}{n-1}\,, \qquad\qquad (4.48)
\end{aligned}$$

where $s_a, \bar{s}_a$ and L_a are integers and $q, \bar{q}, N = 0, 1, \cdots n-2 \mod (n-1)$ are the Z_{n-1} charges of the corresponding fields. Repeating the arguments of section 4.2. we find that

$$w'_- = w_- + \frac{\pi}{\beta_0^2}\left(s_- + \frac{N-2q}{n-1}\right), \qquad \bar{w}'_- = \bar{w}_- + \frac{\pi}{\beta_0^2}\left(\bar{s}_- + \frac{N-2\bar{q}}{n-1}\right) \qquad (4.49)$$

and therefore the b.c.'s for the free field $\Phi_- = 2\beta_0^2\frac{n-1}{n+1}w_-$ (i.e., $J_- = \partial\Phi_-$) are given by

$$\Phi_-(\pm\infty) = \pi\left(s^-_\pm + \frac{N_\pm - 2q_\pm}{n-1}\right) \qquad (4.50)$$

As a consequence the charges Q_0^- and $\bar{Q}_0^-$ (4.20) for the solutions having the above asymptotics take the form,

$$\begin{aligned}
&Q_0^- = 4\pi\frac{n-1}{n+1}\left(s_- + \frac{j^-_w}{n-1}\right), \qquad \bar{Q}_0^- = -4\pi\frac{n-1}{n+1}\left(\bar{s}_- + \frac{j^-_{\bar{w}}}{n-1}\right),\\
&j^-_w = j_\varphi - 2j_q\,, \qquad j_\varphi = N_+ - N_-\,, \qquad s_- = s^-_+ - s^-_-\,, \qquad j_q = q_+ - q_- \qquad (4.51)
\end{aligned}$$

The explicit form $w^-_{\rm top}$ of the U(1)-CFT topological solitons (carriyng $Q_0^-, \bar{Q}_0^-$ topological charges) is again in the standard form (4.40) with $\delta_- = \frac{1}{\beta_0^2}(s_- + \frac{j_\varphi - 2j_q}{n-1})$. Similarly to the u-soliton case (4.45), (4.46) and (4.47) but now according to the charges and energy relations (4.24), (4.27) and (4.28), we derive the following semiclassical spectrum of the left moving 1-soliton of the intermediate IM (2.18):

$$\begin{aligned}
M^2_{\rm int} &= M_g^{p=2}\left(M_g^{p=2} + \left(\frac{16m_0}{\beta_0^2}\right)\frac{n-1}{n+1}\left(s_- + \frac{j_\varphi - 2j_q}{n-1}\right)^2\right),\\
Q^{\rm el}_- &= \frac{\beta_0^2}{2}j_{\rm el}\,, \qquad Q^{\rm el}_+ = \beta_0^2\delta_0\,,\\
Q_{\bar{R}_u} &= 4\frac{n+1}{n-1}\beta_0^2 j_{\rm el} + \pi\left(s_- + \frac{j_\varphi - 2j_q}{n-1}\right),\\
Q^{(1)}_\theta &= -\frac{n+1}{8(n-1)}Q_0^- = -Q^{(n)}_\theta \qquad\qquad (4.52)
\end{aligned}$$

where Q_0^- is given by eq. (4.51). The energy (and the mass $M_{\rm int}(\alpha)$) for more general α-dependent $w^-_{\rm top}(\alpha)$ (see eq. (4.43)) are given by

$$\begin{aligned}
E_{\rm int}(\alpha) &= E_g^{p=2} + E^{\rm int}_{L-{\rm sol}}(\alpha)\,,\\
E^{\rm int}_{L-{\rm sol}}(\alpha) &= \frac{2|a_0|}{\beta_0^2}\frac{n-1}{n+1}\left(\frac{\pi}{\alpha}\right)^2(1-\alpha\coth(\alpha))\left(s_- + \frac{j_\varphi - 2j_q}{n-1}\right)^2,\\
M^2_{\rm int}(\alpha) &= M_g^{p=2}\left(M_g^{p=2} + 2m_0\frac{E^{\rm int}_{L-{\rm sol}}(\alpha)}{|a_0|}\right) \qquad\qquad (4.53)
\end{aligned}$$

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As in the case of the U(1) IM (1.2) (see section 3.5), the intermediate IM (2.18) admits together with the charged topological int-solitons of spectrum (4.52) (or (4.53)) string-like solutions given by eq. (4.26), but with $w^-_{\rm top}(\bar w^-_{\rm top})$ (ref4.37) replaced by

$$w^-_{\rm string} = w^-_{\rm top} + w_{osc}$$

They carry the same topological and electric charges (and new $Q^-_k, \bar Q^-_k, k \in Z$) (4.52), but their energy gets contributions from the oscillator part, i.e.,

$$E^{\rm string}_{\rm int} = E_g + E^{\rm string}_L + E^{\rm string}_R$$

The question of whether such classical string solutions are topologicaly stable (and remain stable under quantization) is still open. The fact that they have the same topological numbers but higher energies, is however an indication for their topological instability.

5. Discussion and further developments

Among the vast variety of integrable perturbations of the gauged A_n-WZW models we have chosen to study the soliton solutions of a specific class of perturbations that preserve one or two chiral U(1) symmetries and whose potentials have n-distinct zeros. Our main tool in the construction of the 1-solitons of these models is the nonconformal version of the GKO coset construction [14], that allows us to compose these u-solitons in terms of massless solitons of the U(1) (or $\mathrm{U}(1)\otimes\mathrm{U}(1)$)-CFT and already known g-solitons of the gauged $G_0/\,\mathrm{U}(1)$ (or $G_0/\,\mathrm{U}(1)\otimes\mathrm{U}(1)$) integrable models [9, 10, 23]. The new ingredients of this construction are the solitons and solitonic strings (left, right and left-right) of the corresponding CFTs presented in section 3 and 4. The simplest example ($n = 1$) of U(1) IMs (1.2) written in the "free field" form

$$\mathcal{L}^{n=1}_u = \partial\Phi\bar\partial\Phi + \beta\bar\partial\gamma + \bar\beta\partial\bar\gamma - \beta\bar\beta e^{-2\Phi} - m^2\gamma\bar\gamma e^{2\Phi} + 2\alpha_0\Phi R^{(2)}\sqrt{-g} \qquad (5.1)$$

(where $R^{(2)}$ is the worldsheet curvature of the 2-d metric $g_{\mu\nu}$, $g = det g_{\mu\nu}$), is known to represent an integrable perturbation of string on AdS_3 target space. One expects that AdS_3/CFT_2 Maldacena correspondence [7] takes place for the relevant deformed theories, i.e. the AdS_3 with perturbation $m^2\gamma\bar\gamma e^{2\Phi}$ to be equivalent to the deformed CFT_2 (i.e. 2-d integrable model) living on the border of the AdS_3 space. Since the energy $\tilde E$ and angular momenta $\tilde l$ of the latter theory are related to the charge Q_0 and $\bar Q_0$ of the AdS_3 model (5.1) [43, 3],

$$Q_0 = \frac{1}{2}(\tilde E + \tilde l)\,, \qquad \bar Q_0 = \frac{1}{2}(\tilde E - \tilde l) \qquad (5.2)$$

the charge spectrum of (5.1) we have derived in section 3 determines the energy spectrum of the border deformed CFT_2. One can consider the $n \geq 2$ IMs (1.2) as specific integrable deformations of the bosonic string on $AdS_3 \otimes T_{n-1}$ target space which provides its charge spectrum (3.36) and (3.52) with certain string (and border CFT_2) meaning. More interesting examples are given by the IMs (2.14) studied in section 4. They can be considered as integrable relevant perturbations of the $AdS_3 \otimes S_3 \otimes T_{n-1}$ string model (taking

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$R_n \to iR_n$ and $\psi_n^* = \chi_n$). It is important to mention that the spectral flow (3.59) of charges $Q_0, \bar{Q}_0$ is realized by adding topological θ-terms (3.55) to the original deformed string lagrangians (1.2).

As it well known the standard procedure of restoring the conformal invariance of the deformed IM (5.1) (and (1.2) in general) is to consider their conformal affine versions [17, 46]. By introducing a new pair of fields (ν, η) one can map the nonconformal IM (5.1) to the following conformal integrable model

$$\begin{aligned}\mathcal{L}_{CAT}^{n=1} &= \partial\Phi\bar{\partial}\Phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \partial\nu\bar{\partial}\eta + \partial\eta\bar{\partial}\nu - \\ &- \beta\bar{\beta}e^{-2\Phi} - m^2\gamma\bar{\gamma}e^{2\Phi-\eta} + 2(\alpha_0\Phi + \gamma_0\nu)R^{(2)}\sqrt{-g} \end{aligned} \tag{5.3}$$

where α_0 and γ_0 are static background charges. An important feature of such IM is that the free field η plays the role of renormalization group parameter that interpolate between different conformal backgrounds (i.e. the zeros of the corresponding σ-model β -functions):

- $\eta \to \infty$ leads back to the original conformal $\mathrm{SL}(2,\mathbb{R})$-WZW model
- $\eta \to 0$ is reproducing the relevant perturbation (5.1)

The model (5.3) admits new massless solitons characterized now by the topological charge

$$Q_\eta = \beta_0 \int_{-\infty}^{\infty} \partial_x \eta dx$$

It is interesting to derive their complete energy and charge spectrum as well as to understand the string meaning of the CAT versions of the solitonic string solutions of (5.1).

An interesting example of an off critical bosonic string on curved target space of black hole type [1] is given by the $n = 2$ intermediate IM (2.18) (with local U(1) symmetry):

$$\begin{aligned}\mathcal{L}_{n=2}^{\mathrm{int}} &= \frac{1}{3}\partial\bar{R}\bar{\partial}\bar{R} + \frac{\sum_{a,b=1}^{2} f_{ab}\left(\left(1 + \frac{\beta^2}{4}\bar{\psi}_a\bar{\chi}_a e^{\beta\epsilon_a\bar{R}}\right)\partial\bar{\chi}_b\bar{\partial}\bar{\psi}_b e^{\beta\epsilon_b\bar{R}} - \bar{\psi}_a\bar{\chi}_b\bar{\partial}\bar{\psi}_b\partial\bar{\chi}_a\right)}{1 + \frac{\beta^2}{4}(\bar{\psi}_1\bar{\chi}_1 e^{\beta\bar{R}} + \bar{\psi}_2\bar{\chi}_2 e^{-\beta\bar{R}})} \\ &- m^2\left(\bar{\psi}_1\bar{\chi}_1 e^{\beta\bar{R}} + \bar{\psi}_2\bar{\chi}_2 e^{-\beta\bar{R}} + \beta^2\bar{\psi}_1\bar{\chi}_1\bar{\psi}_2\bar{\chi}_2\right)\end{aligned} \tag{5.4}$$

where $f_{aa} = 0$, $f_{12} = f_{21} = 1$, $\epsilon_1 = -\epsilon_2 = 1$. For $m^2 = 0$, the σ-model (5.4) is conformal invariant. Since corresponding target space metric admits both horizons and singularities, it represents a specific 5-d generalization of the 3-d black string model [2]. For $m^2 \neq 0$ the above nonconformal IM has U(1)-charged massive and massless solitons (and strings), constructed in section 4.3, that can be interpreted as charged strong coupling (stable) particles (and strings). Its CAT version (i.e. with ν and η as in eqs. (5.3)) might play an important role in the understanding of the nonperturbative properties of the IM (5.4) (and of the corresponding string models) due to the fact that the massless solitons introduces new nonperturbative string states.

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Another direction of extending the results of the present paper is to consider IMs with *nonabelian* local symmetries. The simplest example of integrable perturbation of the SL(3, R)-WZW model that preserve chiral SL(2, $\mathbb{R}$) $\otimes$ U(1) symmetry was derived in our recent paper [23]

$$\mathcal{L}_{\mathrm{SL}(3,R)\ \mathrm{pert}} = \mathcal{L}^{\mathrm{SL}(3,R)}_{WZW}(g_0) - \frac{m^2}{\beta^2}\left(\frac{2}{3} + \psi_1\chi_1 e^{\beta(2\varphi_2-\varphi_1)} + \psi_2\chi_2 e^{\beta(\varphi_1+\varphi_2)}\right) \tag{5.5}$$

where the SL(3, R) group element is parametrized as follows:

$$g_0 = e^{\chi_3 E_{-\alpha_1}} e^{\chi_1 E_{-\alpha_2} + \chi_2 E_{-\alpha_1-\alpha_2}} e^{\varphi_1 h_1 + \varphi_2 h_2} e^{\psi_1 E_{\alpha_2} + \psi_2 E_{\alpha_1+\alpha_2}} e^{\psi_3 E_{\alpha_1}}$$

By the methods of section 2 one can construct integrable perturbations of SL(n + 1)-WZW model keeping unbroken certain subgroup $G_0^0 = \mathrm{SU}(k), k \leq n$. We expect that the nonconformal GKO coset construction of section 3 (for the particular cases $G_0^0 = \mathrm{U}(1)$ or $G_0^0 = \mathrm{U}(1) \otimes \mathrm{U}(1)$) to take place for arbitrary nonabelian G_0^0 as well. As in the U(1) case, the G_0^0-CFT massless solitons should be the main ingredient in the construction of the 1-solitons of such IM, as (5.5) for example. The derivation of their energy and charge spectrum (say for $G_0^0 = \mathrm{SL}(2)$) is an interesting open problem.

Our last comment concerns the nonrelativistic analogs of IM (1.2). The question is whether one can construct *nonrelativistic* IMs with local U(1) symmetry. The well known relation between Lund-Regge (L-R) IM ($n = 1$ of eq. (1.10)) and the nonlinear Schroedinger model [47] and the fact that L-R model is the gauged version of the deformed SL(2, $\mathbb{R}$) WZW (5.1) address the question about the nonrelativistic counterpart (in the sense of ref. [24, 23]) of the IM (5.1). The answer to this question is given by the following integrable system of second order differential equations for the field variables $r(x,t), q(x,t)$ and $u(x,t)$:

$$\begin{aligned}
\partial_t r - \frac{1}{2}\partial_x^2 r + 2u\partial_x r + r\partial_x u - 2r(u^2 - rq) &= 0\,, \\
\partial_t q + \frac{1}{2}\partial_x^2 q + 2u\partial_x q + q\partial_x u + 2q(u^2 - rq) &= 0\,, \\
\partial_t u + \frac{1}{2}\partial_x(rq) &= 0
\end{aligned} \tag{5.6}$$

which is invariant under local U(1) transformations:

$$u = u' + \frac{\beta}{2}\partial_x w(x)\,, \qquad r = e^{\beta w(x)} r'\,, \qquad q = e^{-\beta w(x)} q' \tag{5.7}$$

We expect that its solitons are related to the Non linear Schroedinger (NLS) solitons in the same way the u-solitons of the IM (5.1) are related to the L-R nontopological solitons. The generalization of the IM (5.6) to the family of nonrelativistic IMs with local U(1) symmetry related to the relativistic IMs (1.2) is straightforward.

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